Mechanics of Materials

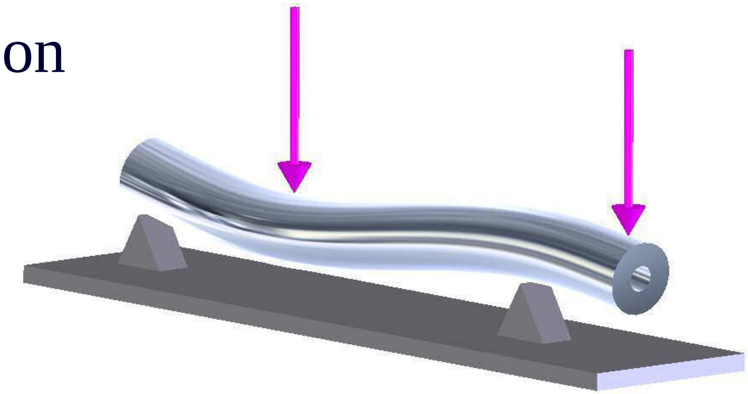
Lecture 11

Deflections of Beams and Shafts (2)

Mohamad Fathi GHANAMEH

Lecture Objectives

- ✓ Use various methods to determine the deflection and slope at specific points on beams and shafts:



  1. Integration method
  2. Discontinuity functions
  3. Method of superposition
  4. Moment-area method

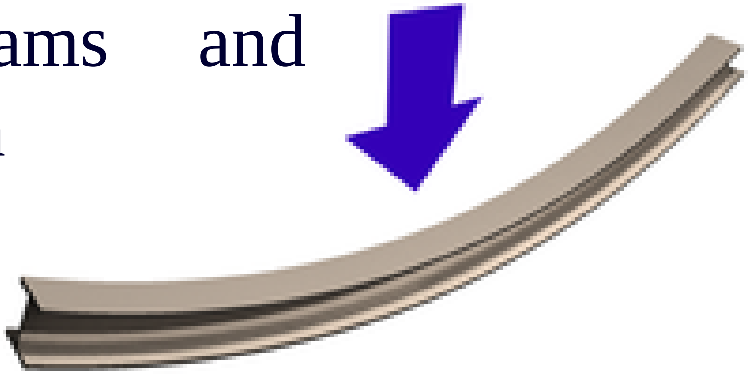
- ✓ Use the various methods to solve for the support reactions on a beam or shaft that is statically indeterminate

Lecture Outline

- ✓ The Elastic Curve
- ✓ Slope and Displacement by Integration
- ✓ Discontinuity Functions
- ✓ Slope and Displacement by the Moment-Area Method
- ✓ Method of Superposition
- ✓ Statically Indeterminate Beams and Shafts
- ✓ Statically Indeterminate Beams and Shafts: Method of Integration

Lecture Outline

- ✓ Statically Indeterminate Beams and Shafts: Moment-Area Method
- ✓ Statically Indeterminate Beams and Shafts: Method of Superposition



Slope and displacement by the moment-area

- Assumptions:
  - beam is initially straight,
  - is elastically deformed by the loads, such that the slope and deflection of the elastic curve are very small, and
  - deformations are caused by bending.

Theorem 1

- The angle between the tangents at any two points on the elastic curve equals the area under the *M/EI* diagram between these two points.

$$\theta_{B/A} = \int_A^B \frac{M}{EI} dx$$

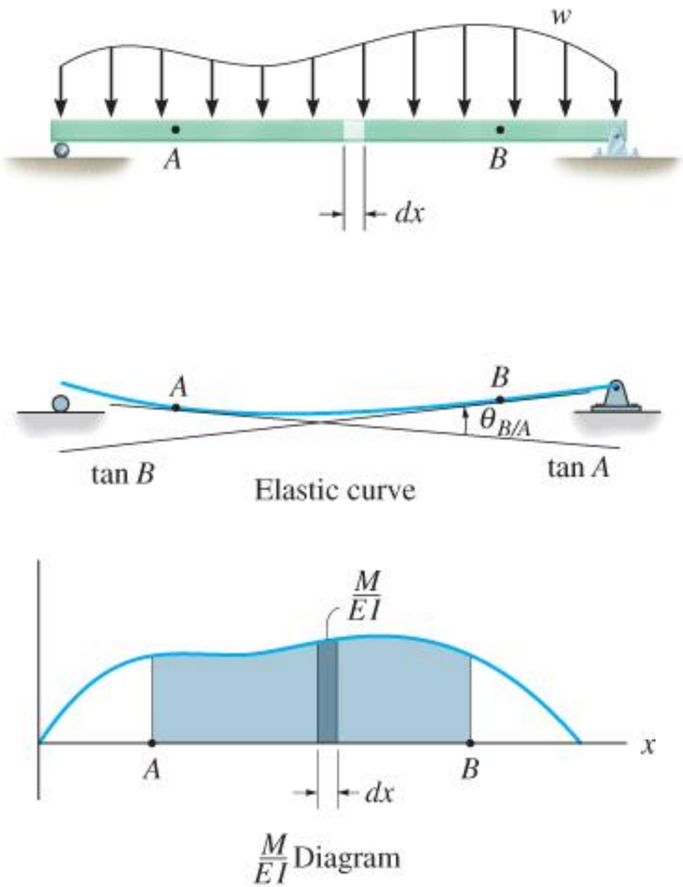
Slope and displacement by the moment-area

Theorem 1

$$M = EI\frac{d^2\upsilon}{dx^2} = EI\frac{d}{dx}\left(\frac{d\upsilon}{dx}\right)$$

$$\Rightarrow d\theta = \frac{M}{EI}dx \Rightarrow \theta_{B/A} = \int_A^B \frac{M}{EI}dx$$

The angle between the tangents at any two points on the elastic curve equals the area under the *M/EI* diagram between these two points.

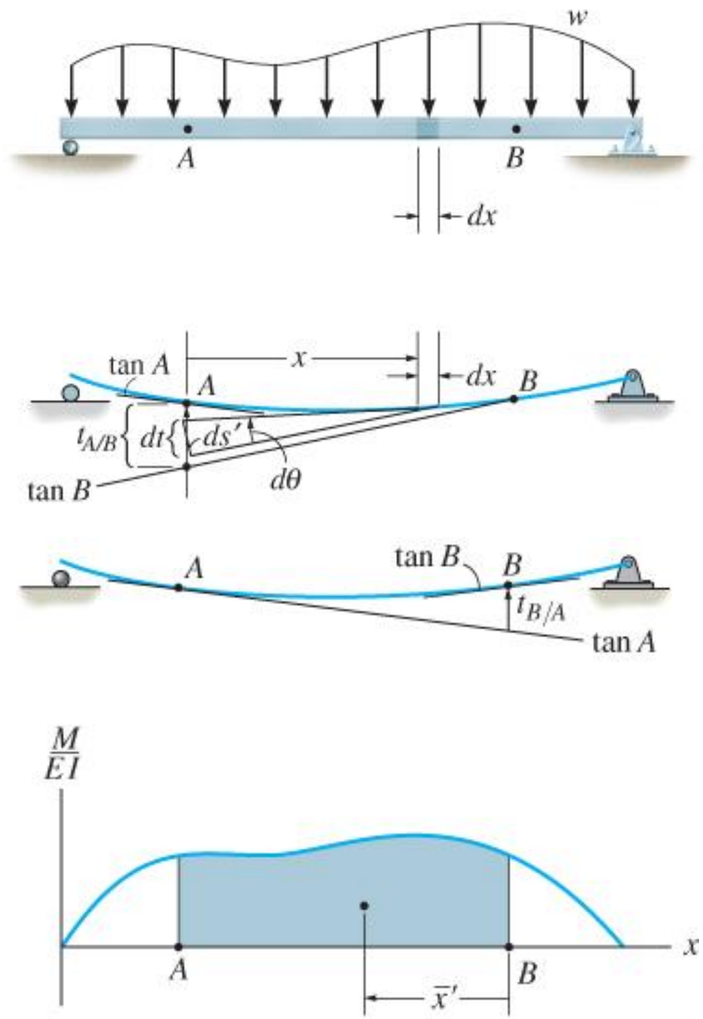


Slope and displacement by the moment-area

Theorem 2

$$s = r.\theta \Leftrightarrow dt = x.\theta$$

$$\left.\begin{array}{l} dt = \int_A^B x \dfrac{M}{EI} dx \\ \bar{x} \int dA = \int x.dA \end{array}\right\} \Rightarrow dt = \bar{x} \int_A^B \frac{M}{EI} dx$$

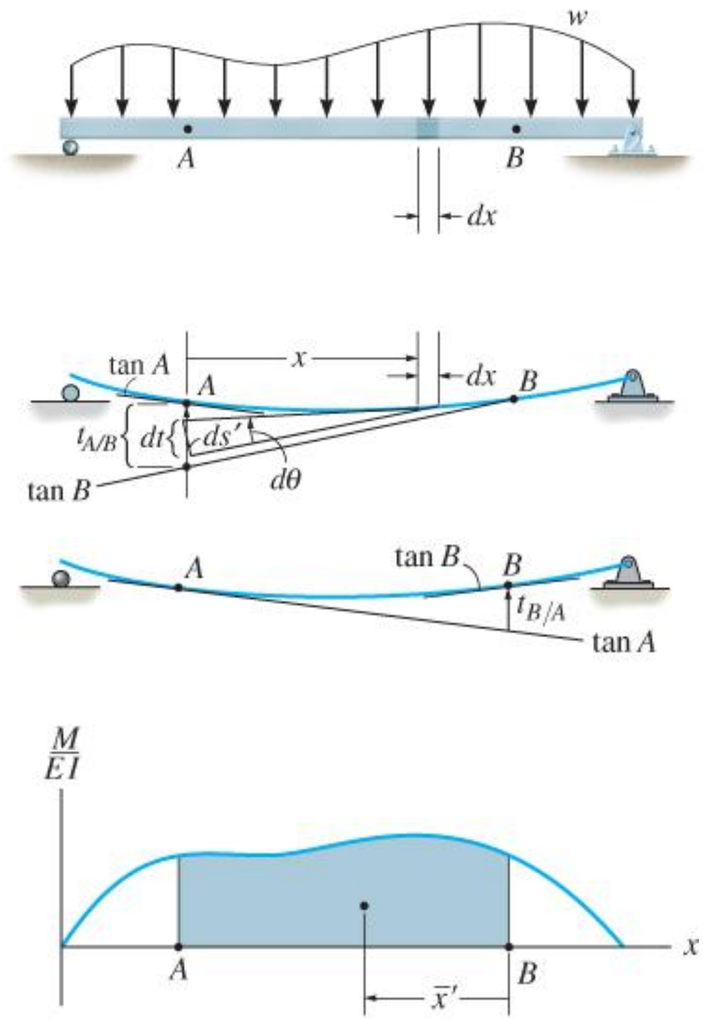


Slope and displacement by the moment-area

Theorem 2

The vertical deviation of the tangent at a point (A) on the elastic curve with respect to the tangent extended from another point (B) equals the moment of the area under the ME/I diagram between these two pts (A and B).

This moment is computed about point (A) where the vertical deviation ($t_{A/B}$) is to be determined.



Slope and displacement by the moment-area

Procedure for analysis

M/ *EI* Diagram

- Determine the support reactions and draw the beam's *M/EI* diagram.
- If the beam is loaded with concentrated forces, the *M/EI* diagram will consist of a series of straight line segments, and the areas and their moments required for the moment-area theorems will be relatively easy to compute.
- If the loading consists of a series of distributed loads, the *M/EI* diagram will consist of parabolic or perhaps higher-order curves, and we use the table on the inside front cover to locate the area and centroid under each curve.

Slope and displacement by the moment-area

Procedure for analysis

Elastic curve

- Draw an exaggerated view of the beam's elastic curve.
- Recall that pts of zero slope and zero displacement always occur at a fixed support, and zero displacement occurs at all pin and roller supports.
- If it is difficult to draw the general shape of the elastic curve, use the moment (M/EI) diagram.
- Realize that when the beam is subjected to a positive moment, the beam bends concave up, whereas -ve moment bends the beam concave down.

Slope and displacement by the moment-area

Procedure for analysis

Elastic curve

- An inflection point or change in curvature occurs when the moment if the beam (or M/EI) is zero.
- The unknown displacement and slope to be determined should be indicated on the curve.
- Since moment-area theorems apply only between two tangents, attention should be given as to which tangents should be constructed so that the angles or deviations between them will lead to the solution of the problem.
- The tangents at the supports should be considered, since the beam usually has zero displacement and/or zero slope at the supports.

Slope and displacement by the moment-area

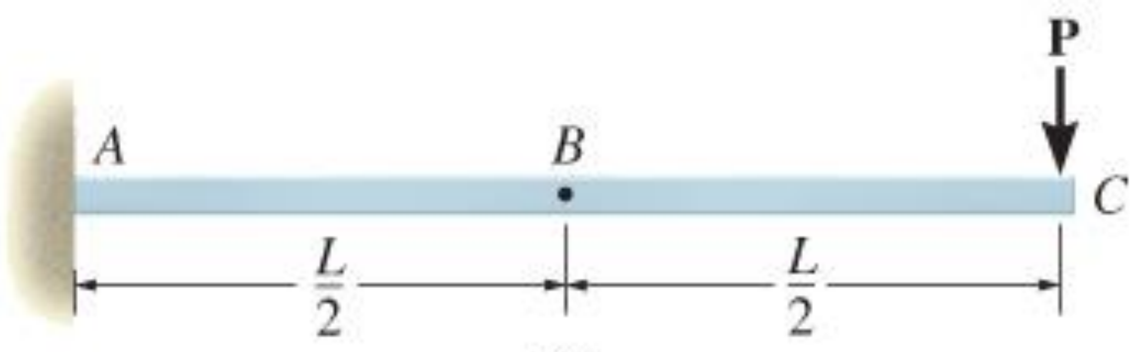
Procedure for analysis

Moment-area theorems

- Apply Theorem 1 to determine the angle between any two tangents on the elastic curve and Theorem 2 to determine the tangential deviation.
- The algebraic sign of the answer can be checked from the angle or deviation indicated on the elastic curve.
- A positive $\theta_{B/A}$ represents a counterclockwise rotation of the tangent at B w.r.t. tangent at A, and a positive $t_{B/A}$ indicates that point B on the elastic curve lies above the extended tangent from point A.

Slope and displacement by the moment-area

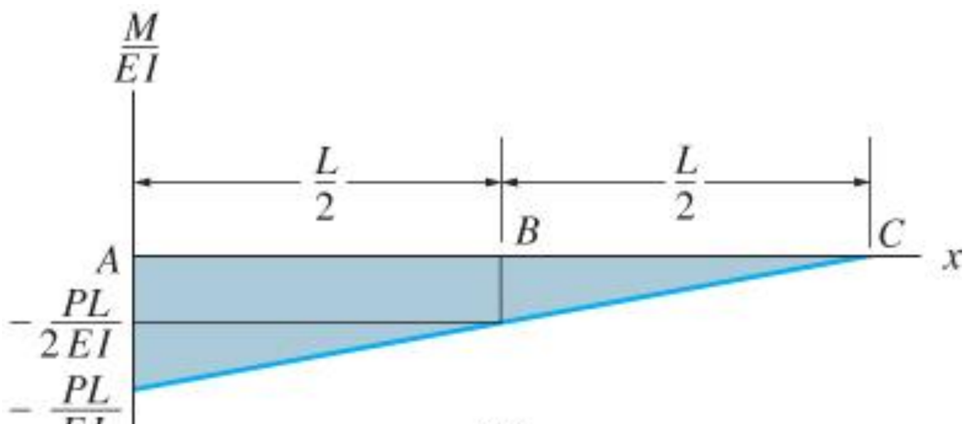
Determine the slope of the beam shown at pts *B* and *C*. *EI* is constant.



(a)

Slope and displacement by the moment-area

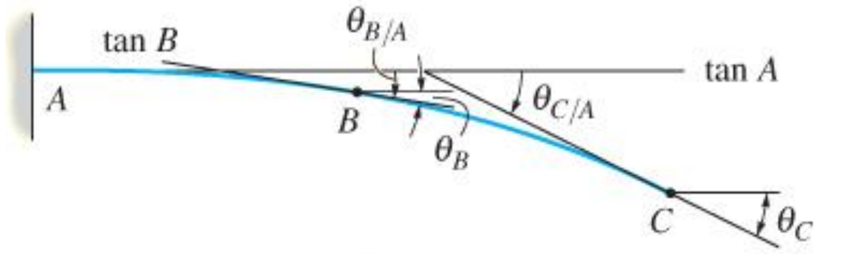
M/EI diagram:



(b)

Elastic curve:

The force **P** causes the beam to deflect as shown.



(c)

Slope and displacement by the moment-area

Elastic curve:

The tangents at B and C are indicated since we are required to find B and C. Also, the tangent at the support (A) is shown. This tangent has a known zero slope. By construction, the angle between tan A and tan B, $\theta_{B/A}$, is equivalent to θ_B, or

$$\theta_B = \theta_{B/A} \quad \text{and} \quad \theta_C = \theta_{C/A}$$

Slope and displacement by the moment-area

Moment-area theorem:

Applying Theorem 1, $\theta_{B/A}$ is equal to the area under the *M/EI* diagram between pts *A* and *B*, that is,

$$\theta_B = \theta_{B/A} = \left(-\frac{PL}{2EI}\right)\left(\frac{L}{2}\right) + \frac{1}{2}\left(-\frac{PL}{2EI}\right)\left(\frac{L}{2}\right)$$

$$= -\frac{3PL^2}{8EI}$$

Slope and displacement by the moment-area

Moment-area theorem:

The negative sign indicates that angle measured from tangent at *A* to tangent at *B* is clockwise. This checks, since beam slopes downward at *B*.

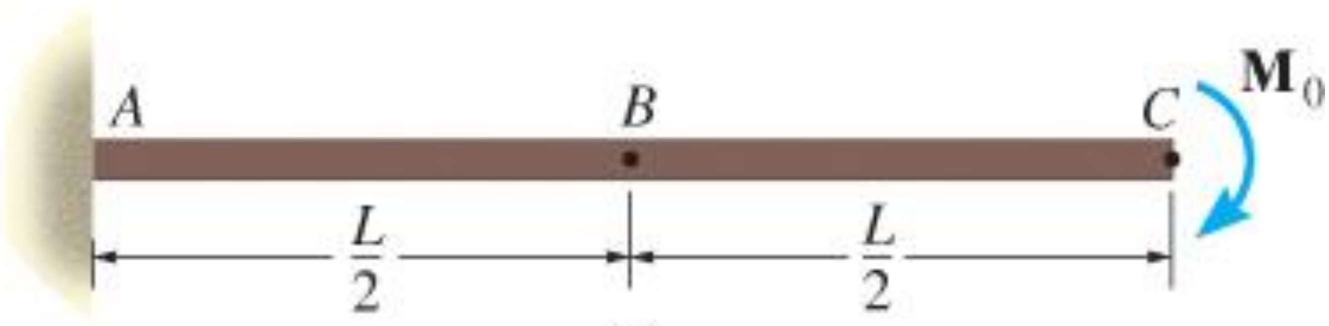
Similarly, area under the *M/EI* diagram between pts *A* and *C* equals $\theta_{C/A}$. We have

$$\theta_C = \theta_{C/A} = \frac{1}{2}\left(-\frac{PL}{EI}\right)L$$

$$= -\frac{PL^2}{2EI}$$

Slope and displacement by the moment-area

Determine the displacement of pts *B* and *C* of beam shown. *EI* is constant.



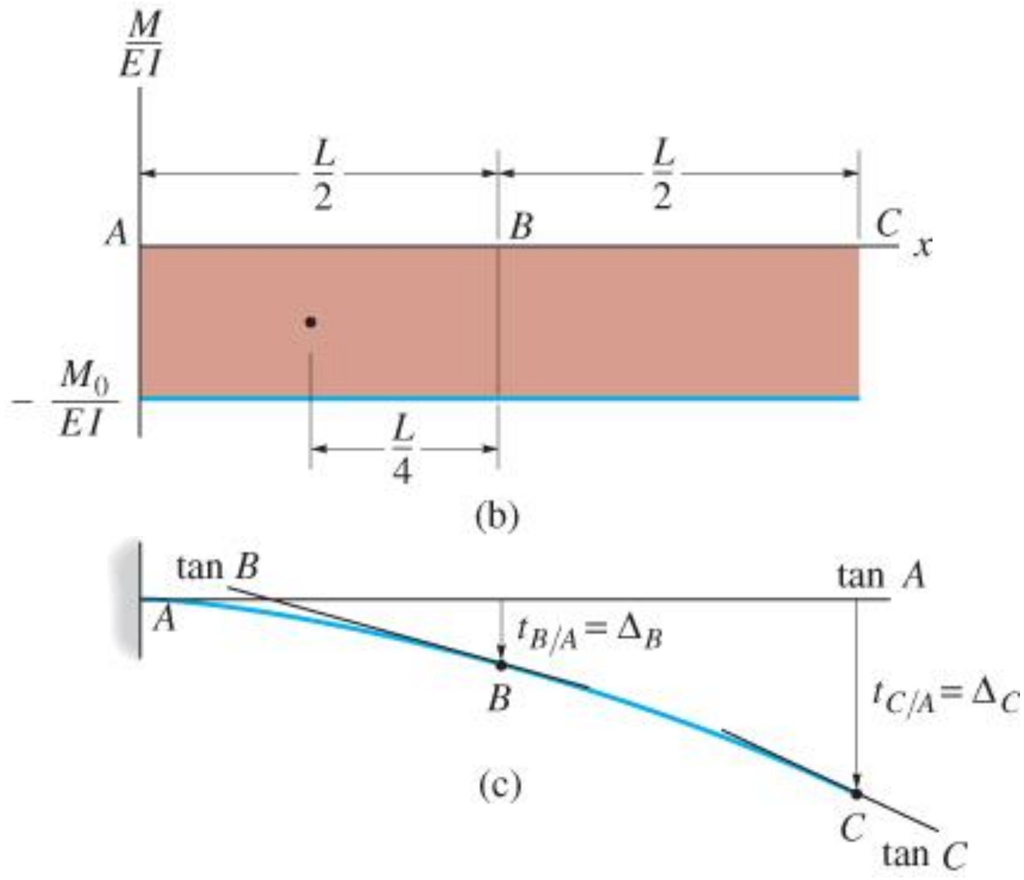
(a)

Slope and displacement by the moment-area

M/EI diagram:

Elastic curve:

The couple moment at C cause the beam to deflect as shown.

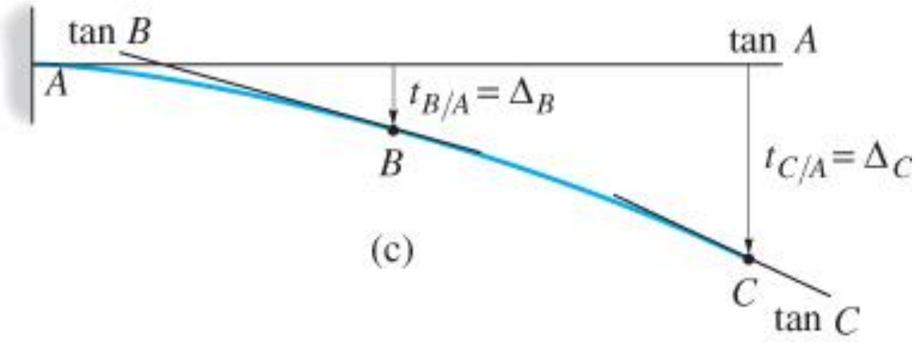


Slope and displacement by the moment-area

Elastic curve:

The required displacements can be related directly to deviations between the tangents at *B* and *A* and *C* and *A*. Specifically, Δ_B is equal to deviation of tan *A* from tan *B*,

$$\Delta_B = t_{B/A} \qquad \Delta_C = t_{C/A}$$



Slope and displacement by the moment-area

Moment-area theorem:

Applying Theorem 2, $t_{B/A}$ is equal to the moment of the shaded area under the *M/EI* diagram between *A* and *B* computed about point *B*, since this is the point where tangential deviation is to be determined. Hence,

$$\Delta_B = t_{B/A} = \left(\frac{L}{4}\right)\left[\left(-\frac{M_0}{EI}\right)\left(\frac{L}{2}\right)\right] = -\frac{M_0 L^2}{8EI}$$

Slope and displacement by the moment-area

Moment-area theorem:

Likewise, for $t_{C/A}$ we must determine the moment of the area under the entire *M*/*EI* diagram from *A* to *C* about point *C*. We have

$$\Delta_C = t_{C/A} = \left(\frac{L}{2}\right)\left[\left(-\frac{M_0}{EI}\right)(L)\right] = -\frac{M_0 L^2}{2EI}$$

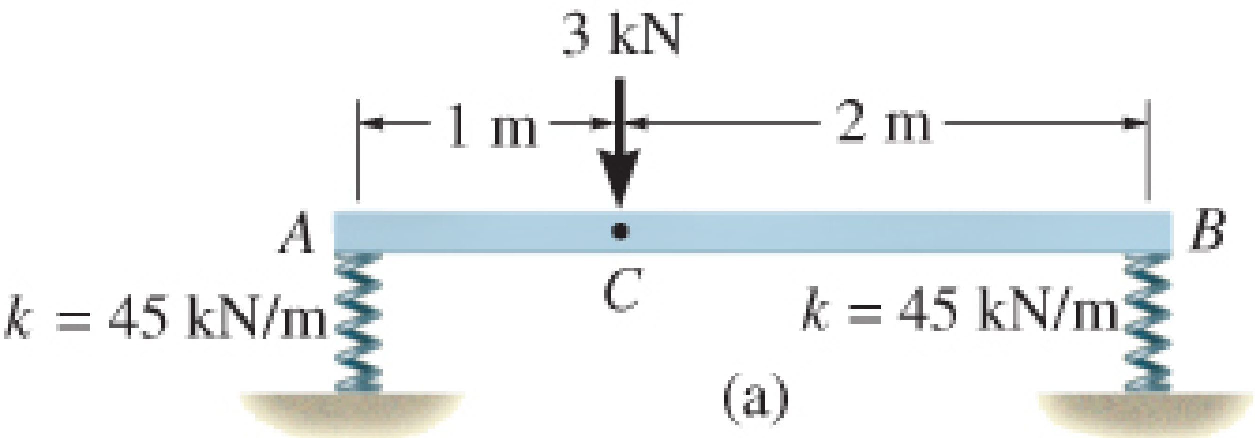
Since both answers are –ve, they indicate that pts *B* and *C* lie below the tangent at *A*. This checks with the figure.

Slope and displacement by method of superposition

- The differential Equation $EI\, d^4v/dx^4 = -w(x)$ satisfies the two necessary requirements for applying the principle of superposition
- The load $w(x)$ is linearly related to the deflection $v(x)$
- The load is assumed not to change significantly the original geometry of the beam or shaft.

Slope and displacement by method of superposition

Steel bar shown is supported by two springs at its ends *A* and *B*. Each spring has a stiffness k = 45 kN/m and is originally unstretched. If the bar is loaded with a force of 3 kN at point *C*, determine the vertical displacement of the force. Neglect the weight of the bar and take E_{st} = 200 GPa, $I = 4.6875 \times 10^{-6}$ m.

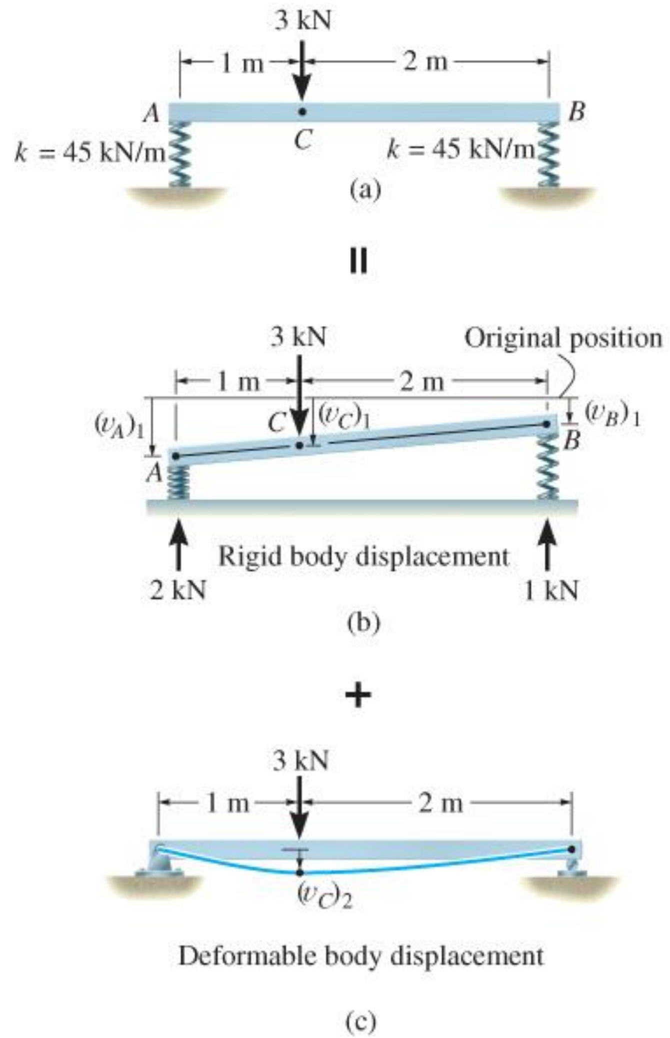


Slope and displacement by method of superposition

End reactions at *A* and *B* are computed and shown. Each spring deflects by an amount

$$(\upsilon_A)_1 = \frac{2\ \text{kN}}{45\ \text{kN/m}} = 0.0444\ \text{m}$$

$$(\upsilon_B)_1 = \frac{1\ \text{kN}}{45\ \text{kN/m}} = 0.0222\ \text{m}$$



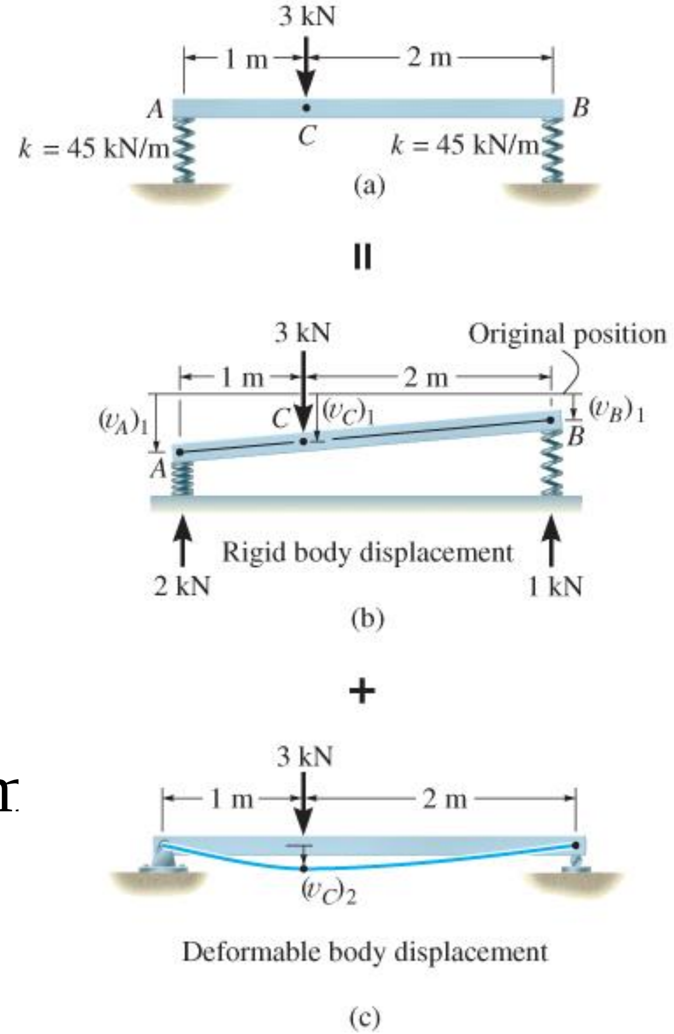
Slope and displacement by method of superposition

If bar is considered rigid, these displacements cause it to move into positions shown. For this case, the vertical displacement at *C* is

$$(\upsilon_C)_1 = (\upsilon_B)_1 + \frac{2\text{ m}}{3\text{ m}}\left[(\upsilon_A)_1 - (\upsilon_B)_1\right]$$

$$= 0.0222\text{ m} + \frac{2}{3}[0.0444\text{ m} - 0.0282\text{ m}]$$

$$= 0.0370\text{ m} \downarrow$$



Slope and displacement by method of superposition

We can find the displacement at C caused by the deformation of the bar.

We have

$$(\upsilon_C)_2 = \frac{Pab}{6EIL}\left(L^2 - b^2 - a^2\right)$$

$$= \frac{(3\text{ kN})(1\text{ m})(2\text{ m})\left[(3\text{ m})^2 - (2\text{ m})^2 - (1\text{ m})^2\right]}{6(200)\left(10^6\right)\text{kN/m}^2(4.6875)\left(10^{-6}\right)\text{m}^4(3\text{ m})}$$

$$= 1.422\text{ mm} \downarrow$$

Slope and displacement by method of superposition

Adding the two displacement components, we get

$$(+\downarrow) \quad \upsilon_C = 0.0370\text{ m} + 0.001422\text{ m}$$

$$= 0.0384\text{ m} = 38.4\text{ mm}$$

Statically Indeterminate Beams and Shafts

- A member of any type is classified as statically indeterminate if the no. of unknown reactions exceeds the available no. of equilibrium eqns.
- Additional support reactions on beam that are not needed to keep it in stable equilibrium are called redundant.
- No. of these redundant is referred to as the degree of indeterminacy.
- To determine the reactions, it is necessary to specify the redundant reactions. that can be determined using compatibility conditions.

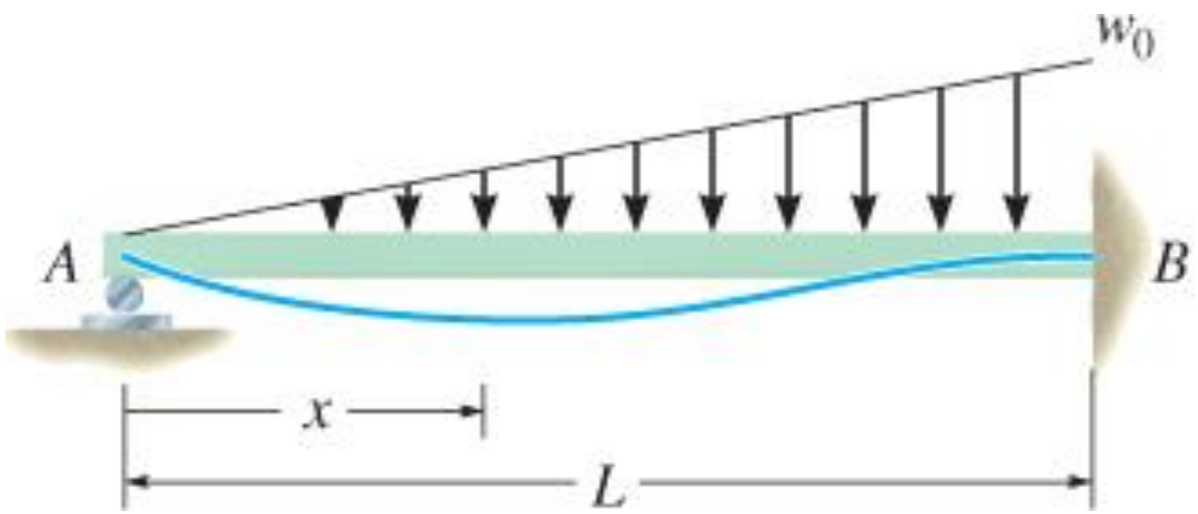
Statically Indeterminate Beams and Shafts

Method of Integration:

- For a statically indeterminate beam, the internal moment M can be expressed in terms of the unknown redundants.
- After integrating this Equation twice, there will be two constants of integration and the redundants to be found.
- The unknowns can be found from the boundary and/or continuity conditions for the problem.

Method of Integration– Example

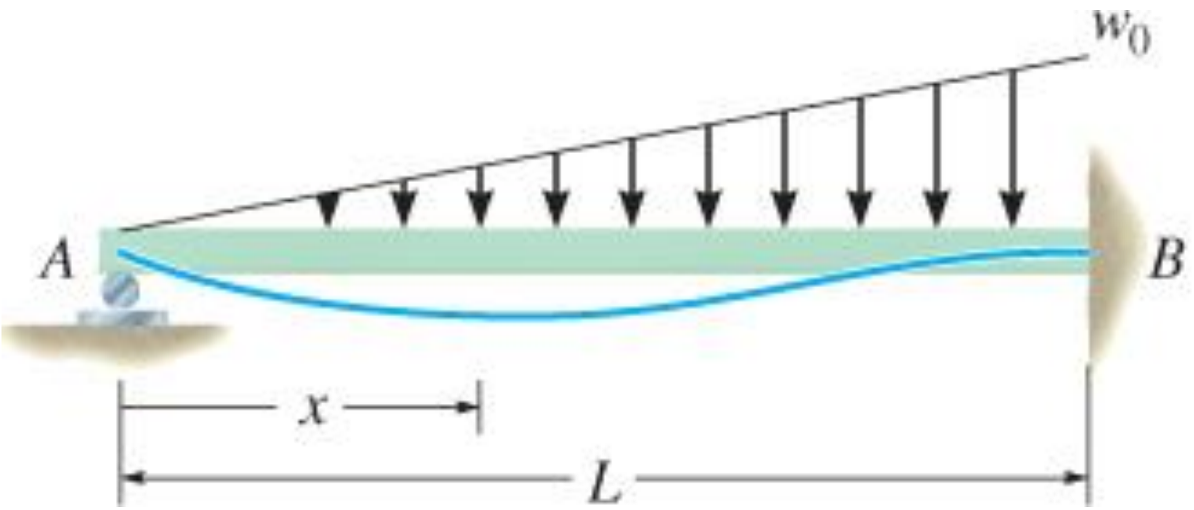
Beam is subjected to the distributed loading shown. Determine the reactions at *A*. *EI* is a constant.



Method of Integration– Example

Elastic curve:

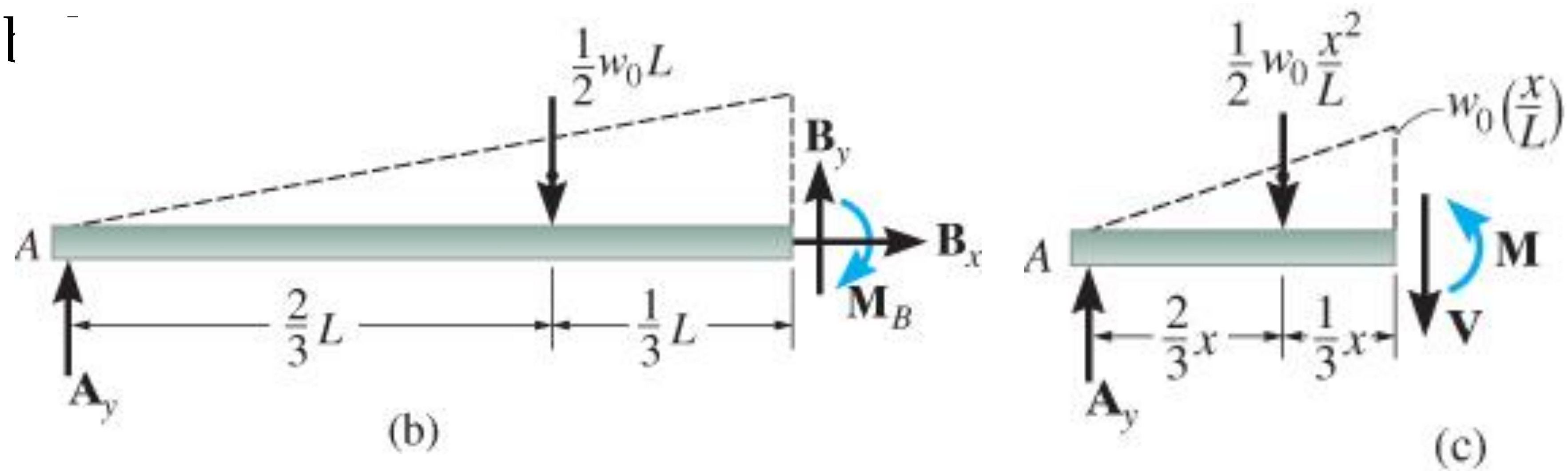
Beam deflects as shown. Only one coordinate *x* is needed. For convenience, we will take it directed to the right, since internal moment is easy to formulate.



Method of Integration– Example

Moment function:

Beam is indeterminate to first degree as indicated from the free-body diagram. We can express the internal moment *M* in terms of the redundant force at A using segment shown



Statically Indeterminate Beams and Shafts

Method of Integration– Example

Moment function:

$$M = A_y x - \frac{1}{6} w_0 \frac{x^3}{L}$$

Slope and elastic curve:

$$EI \frac{d^2\upsilon}{dx^2} = A_y x - \frac{1}{6} w_0 \frac{x^3}{L}$$

$$EI \frac{d\upsilon}{dx} = \frac{1}{2} A_y x^2 - \frac{1}{24} w_0 \frac{x^4}{L} + C_1$$

$$EI\upsilon = \frac{1}{6} A_y x^3 - \frac{1}{120} w_0 \frac{x^5}{L} + C_1 x + C_2$$

Method of Integration– Example

Slope and elastic curve:

The three unknowns A_y, C_1 and C_2 are determined from the boundary conditions $x = 0$, $v = 0$; $x = L$, $dv/dx = 0$; and $x = L$, $v = 0$. Applying these conditions yields

$$x = 0,\ \upsilon = 0; \qquad 0 = 0 - 0 + 0 + C_2$$

$$x = L,\ \frac{d\upsilon}{dx} = 0; \qquad 0 = \frac{1}{2}A_y L^2 - \frac{1}{24}w_0 L^3 + C_1$$

$$x = L,\ \upsilon = 0; \qquad 0 = \frac{1}{6}A_y L^3 - \frac{1}{120}w_0 L^4 + C_1 L + C_2$$

Method of Integration– Example

Slope and elastic curve:

Solving,

$$A_y = \frac{1}{10} w_0 L$$

$$C_1 = -\frac{1}{120} w_0 L^3 \qquad C_2 = 0$$

Using the result for A_y, the reactions at B can be determined from the equations of equilibrium. Show that $B_x = 0$. $B_y = 2 w_0 L/5$ and $M_B = w_0 L^2/15$

Statically Indeterminate Beams and Shafts

Moment-area

- Draw the *M/EI* diagrams such that the redundants are represented as unknowns.
- Apply the 2 moment-area theorems to get the relationships between the tangents on elastic curve to meet conditions of displacement and/or slope at supports of beam.
- For all cases, no. of compatibility conditions is equivalent to no. of redundants.

Statically Indeterminate Beams and Shafts

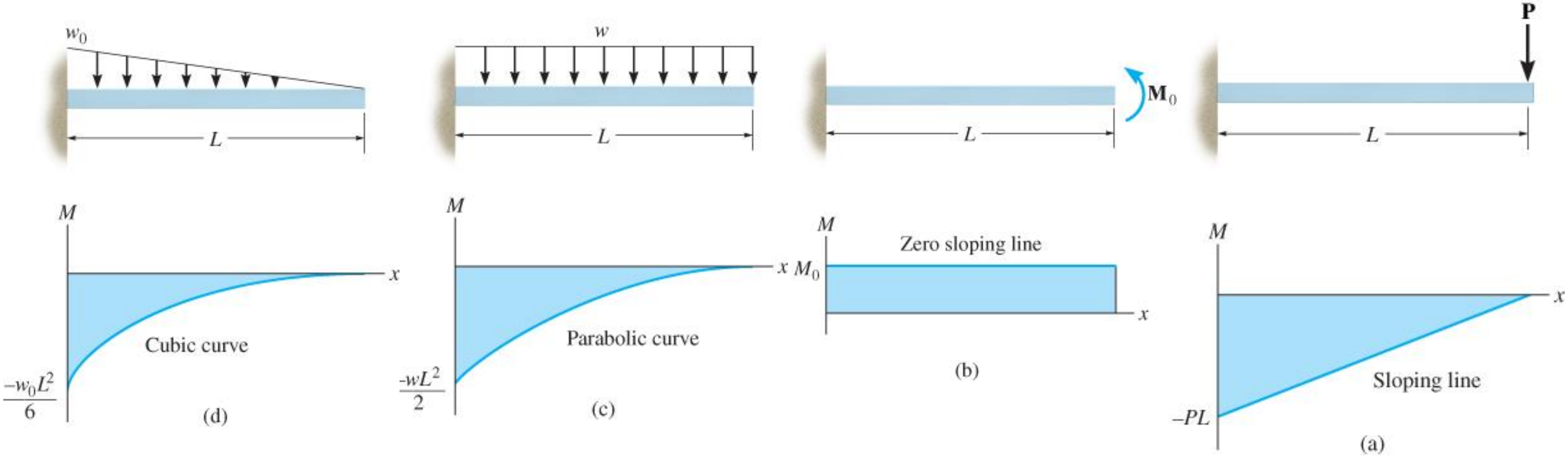
Moment diagrams using superposition method

- Since moment-area theorems needs calculation of both the area under the M/EI diagram and centroidal location of this area, the method of superposition can be used to combine separate M/EI diagrams for each of the known loads.
- This will be relevant if the resultant moment diagram is of a complicated shape.

Statically Indeterminate Beams and Shafts

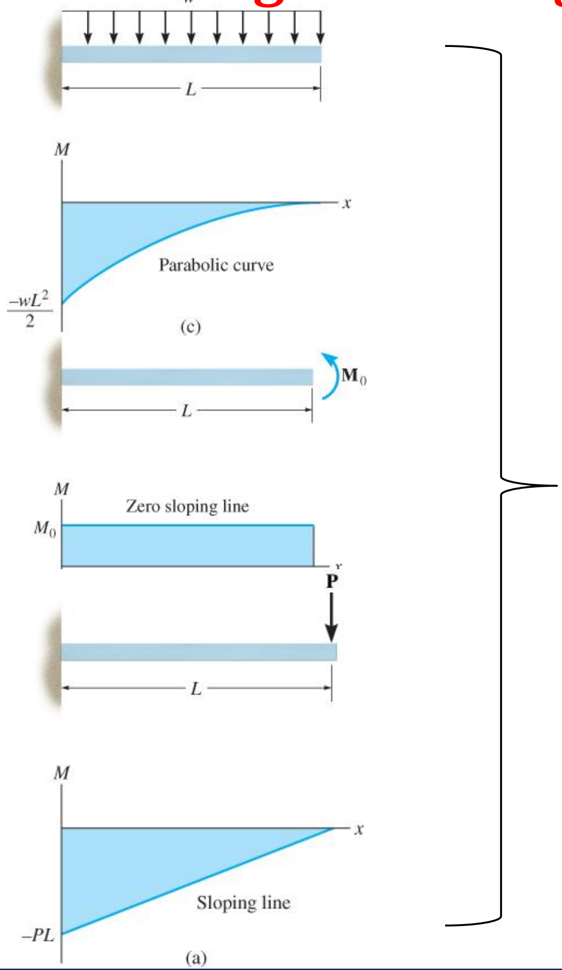
Moment diagrams using superposition method

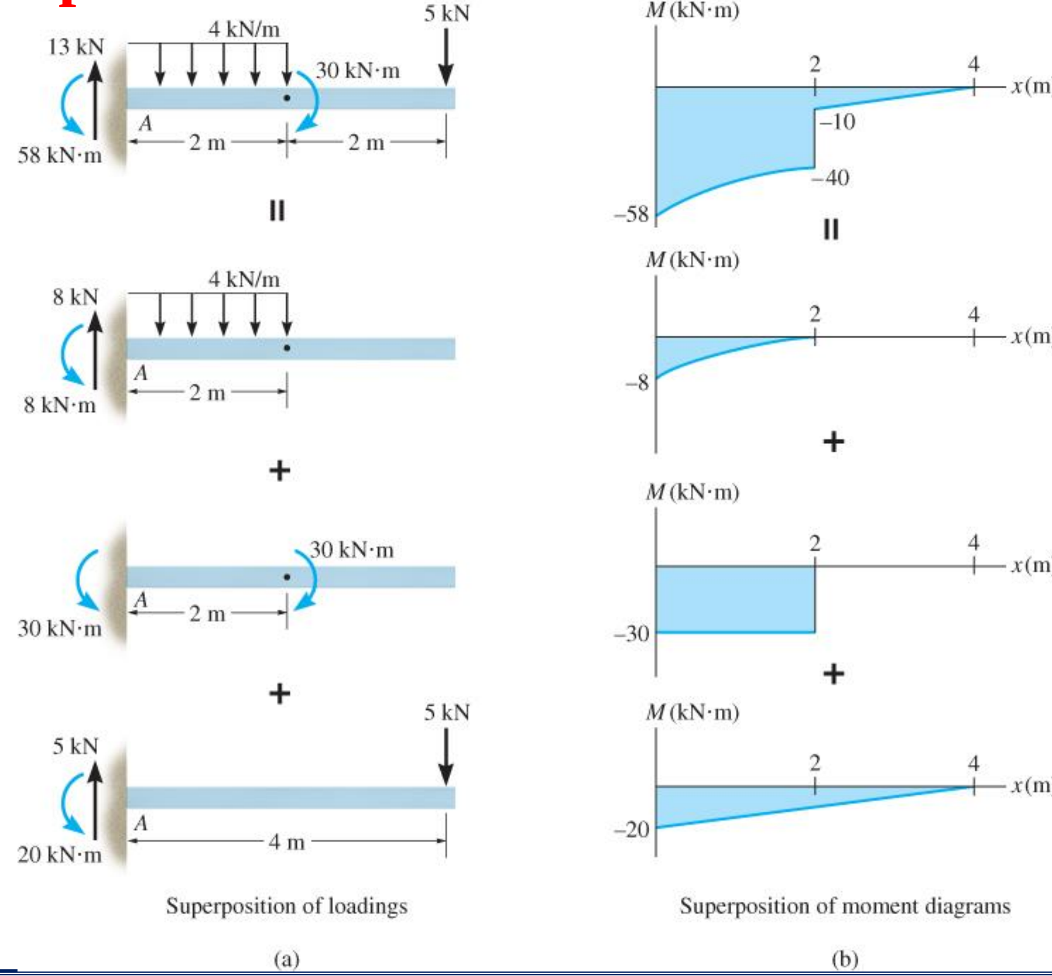
- Most loadings on beams are a combination of the four loadings as shown.



Statically Indeterminate Beams and Shafts

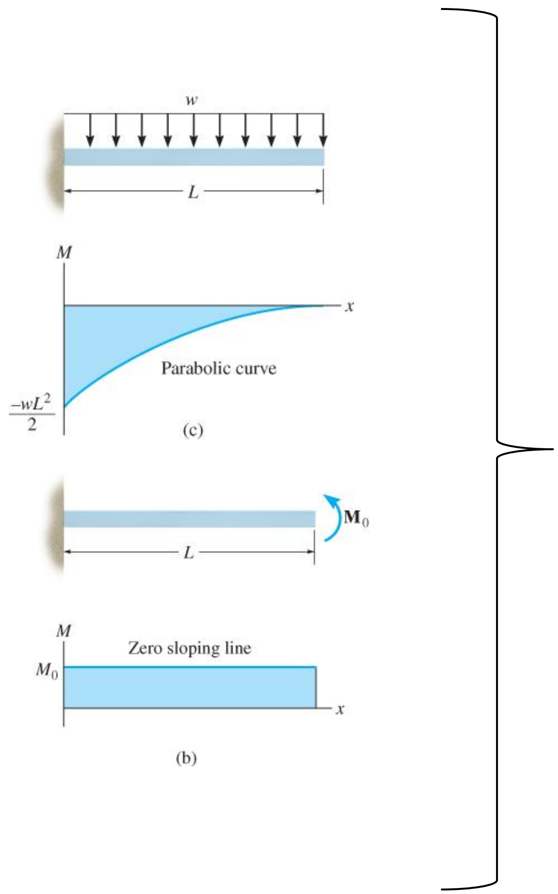
Moment diagrams using superposition method

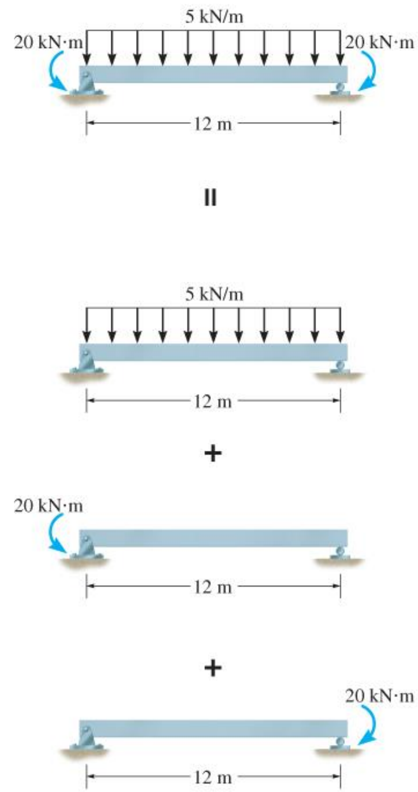




Statically Indeterminate Beams and Shafts

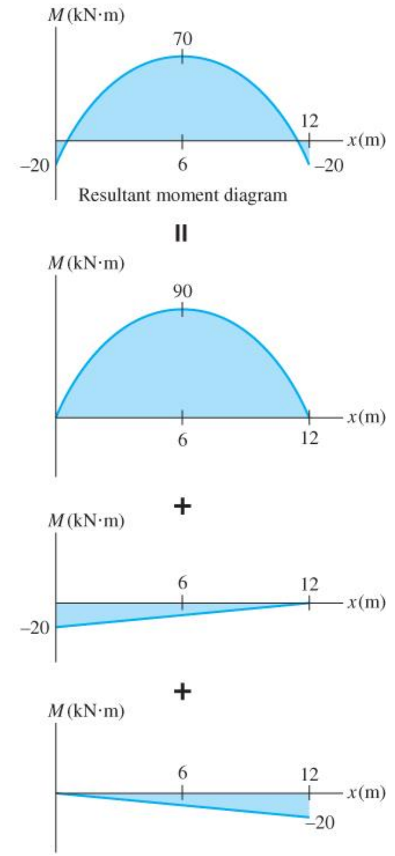
Moment diagrams using superposition method





Superposition of loadings

(a)

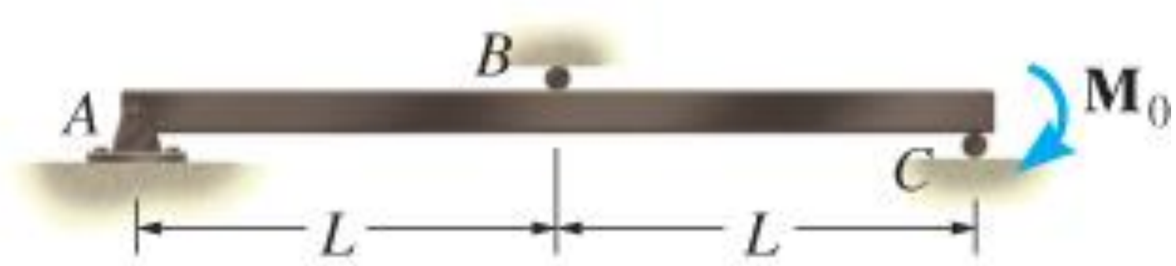


Superposition of moment diagrams

(b)

Moment diagrams using superposition method - Example

Beam is subjected to couple moment at its end *C* as shown. Determine the reaction at *B*. *EI* is constant.

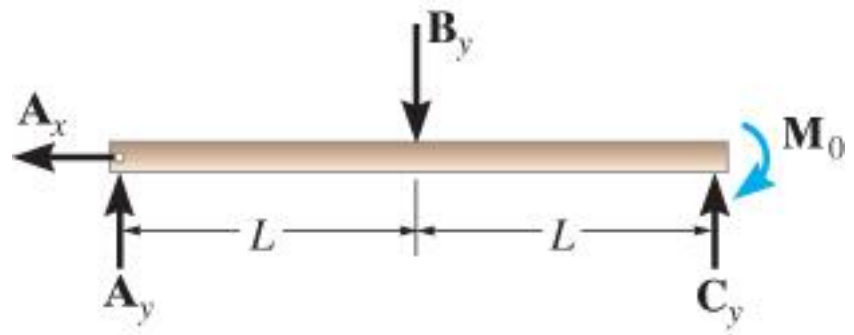


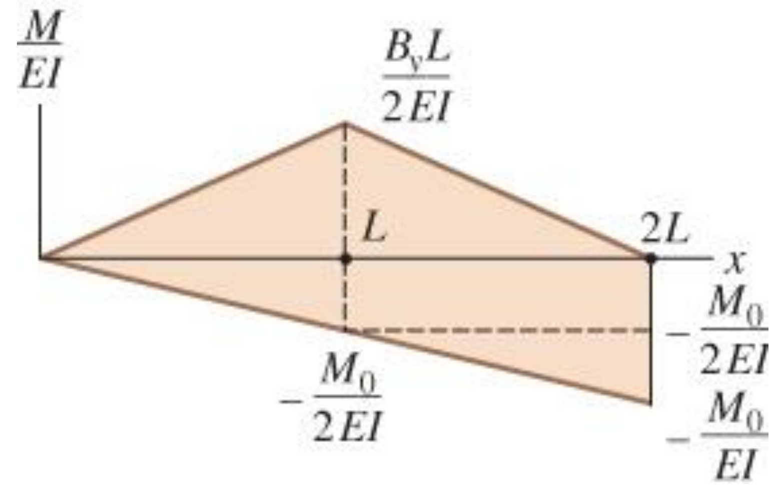
Statically Indeterminate Beams and Shafts

M/EI Diagram:

Free-body diagram as shown. By inspection, beam is indeterminate to first degree. To get a direct solution, choose $\mathbf{B}_y$ as the redundant.

Using superposition, the *M/EI* diagrams for $\mathbf{B}_y$ and $\mathbf{M}_0$, each applied to the simply supported beam are shown.

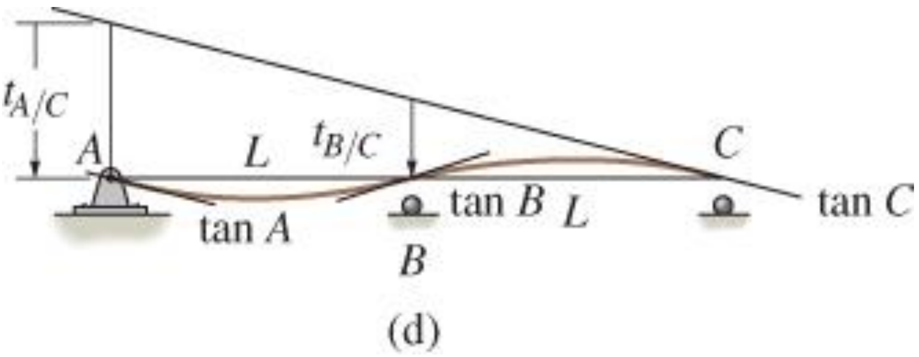




Statically Indeterminate Beams and Shafts

Elastic curve:

Elastic curve as shown. Tangents at *A*, *B* and *C* has been established.



Since $\Delta_A = \Delta_B = \Delta_C = 0$, then tangential deviations shown must be proportional,

$$t_{B/C} = \frac{1}{2} t_{A/C} \qquad (1)$$

Statically Indeterminate Beams and Shafts

Elastic curve:

From *ME/I* diagram, we have

$$t_{B/C} = \left(\frac{1}{3}L\right)\left[\frac{1}{2}\left(\frac{B_y L}{2EI}\right)(L)\right] + \left(\frac{2}{3}L\right)\left[\frac{1}{2}\left(\frac{-M_0}{2EI}\right)(L)\right] + \left(\frac{L}{2}\right)\left[\left(\frac{-M_0}{2EI}\right)(L)\right]$$

$$t_{A/C} = (L)\left[\frac{1}{2}\left(\frac{B_y L}{2EI}\right)(2L)\right] + \left(\frac{2}{3}(2L)\right)\left[\frac{1}{2}\left(\frac{-M_0}{EI}\right)(2L)\right]$$

Statically Indeterminate Beams and Shafts

Elastic curve:

Substituting into Equation(1), we have

$$B_y = \frac{3M_0}{2L}$$

Equations of equilibrium:

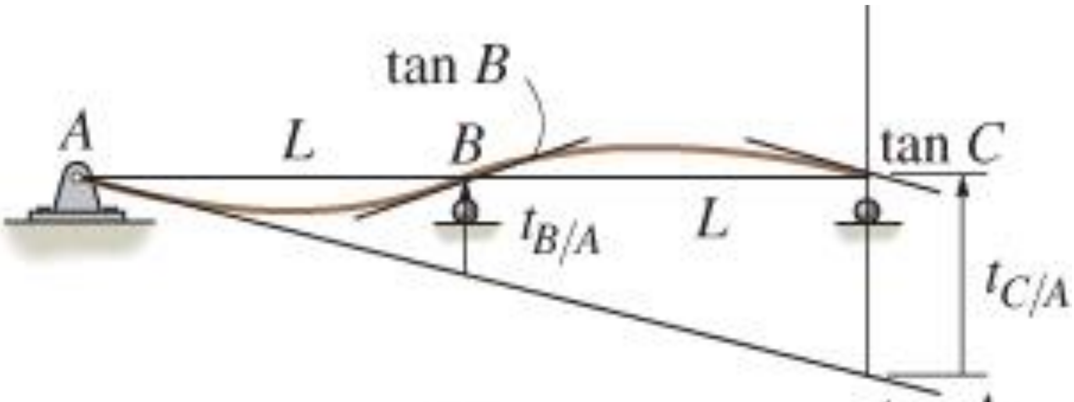
Reactions at *A* and *C* can now be determined from the eqns of equilibrium. Show that $A_x = 0$, $C_y = 5M_0/4L$, and $A_y = M_0/4L$.

Statically Indeterminate Beams and Shafts

Equations of equilibrium:

From figure shown, this problem can also be worked out in terms of the tangential deviations,

$$t_{B/A} = \frac{1}{2} t_{C/A}$$



(e)

Statically Indeterminate Beams and Shafts

- First, identify the redundant support reactions on the beam.
- Remove these reactions from the beam to get a primary beam that is statically determinate and stable and subjected to external load only.
- Add to this beam with a series of similarly supported beams, each with a separate redundant, then by principle of superposition, the final loaded beam is obtained.
- After computing the redundants, the other reactions on the beam determined from the equations of equilibrium.
- This method of analysis is sometimes called the force method.

Statically Indeterminate Beams and Shafts

Procedure for analysis

Elastic curve

- Specify unknown redundant forces or moments that must be removed from the beam in order to make it statically determinate and stable.
- Use principle of superposition, draw the statically indeterminate beam and show it to be equal to a sequence of corresponding statically determinate beams.
- The first beam (primary) supports the same external loads as the statically indeterminate beam, and each of the other beams “added” to the primary beam shows the beam loaded with a separate single redundant force or moment.

Statically Indeterminate Beams and Shafts

Procedure for analysis

Elastic curve

- Sketch the deflection curve for each beam and indicate symbolically the displacement or slope at the point of each redundant force or moment.

Compatibility equations

- Write a compatibility Equation for the displacement or slope at each point where there is a redundant force or moment.
- Determine all the displacements or slopes using an appropriate method.
- Substitute the results into the compatibility Equations and solve for the unknown redundants.

Statically Indeterminate Beams and Shafts

Procedure for analysis

Compatibility equations

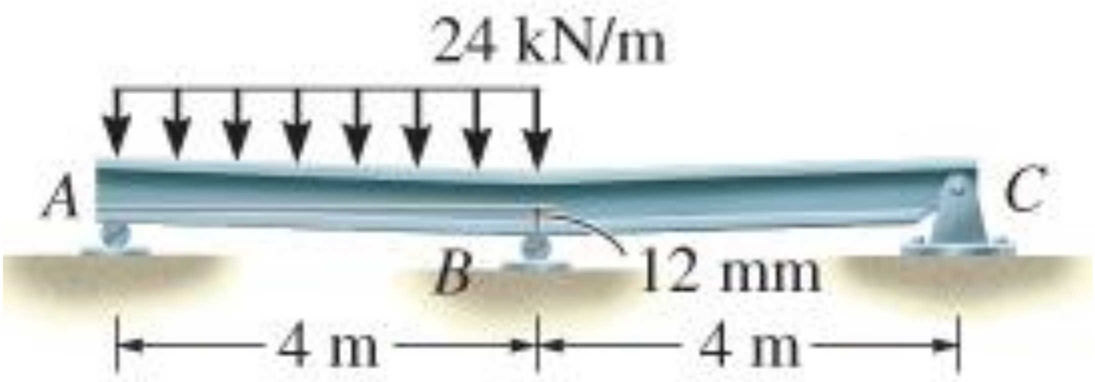
- If a numerical value for a redundant is positive, it has the same sense of direction as originally assumed.
- Similarly, a negative numerical value indicates the redundant acts opposite to its assumed sense of direction.

Equilibrium equations

- Once the redundant forces and/or moments have been determined, the remaining unknown reactions can be found from the equations of equilibrium applied to the loadings shown on the beam’s free-body diagram.

Statically Indeterminate Beams and Shafts

Determine the reactions on the beam shown. Due to loading and poor construction, the roller support at *B* settles 12 mm. Take $E = 200$ GPa and $I = 80(10^6)$ mm^4.



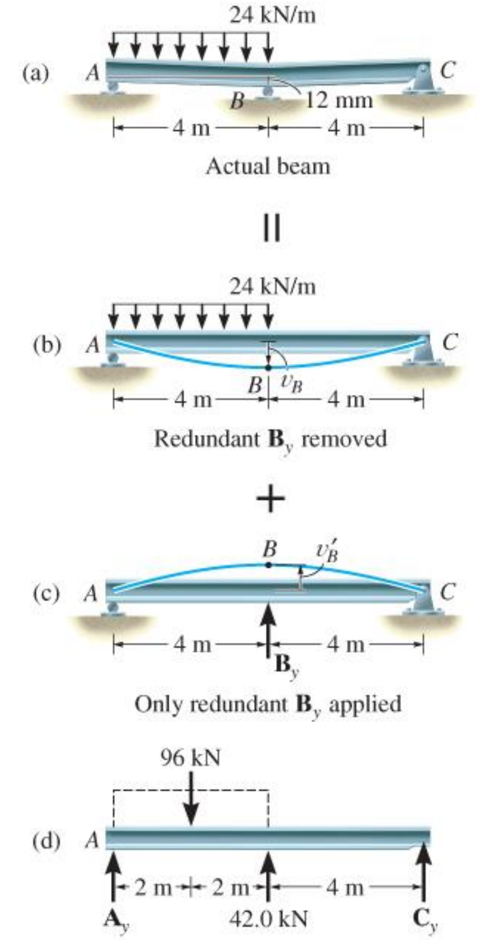
Statically Indeterminate Beams and Shafts

Principle of superposition

By inspection, beam is indeterminate to the first degree. Roller support at *B* is chosen as the redundant.

Principle of superposition is shown.

Here, $\mathbf{B}_y$ is assumed to act upwards on the beam.



Statically Indeterminate Beams and Shafts

Compatibility equation

With reference to point *B*, we require

$$(+\downarrow) \qquad 0.012\ \text{m} = \upsilon_B - \upsilon'_B \qquad (1)$$

Using table in Appendix C, displacements are

$$\upsilon_B = \frac{5wL^4}{768EI} = \frac{5(24\ \text{kN/m})(8\ \text{m})^4}{768EI} = \frac{640\ \text{kN}\cdot\text{m}^3}{EI} \downarrow$$

$$\upsilon'_B = \frac{PL^3}{48EI} = \frac{B_y(8\ \text{m})^3}{48EI} = \frac{10.67\ \text{m}^3 B_y}{EI} \uparrow$$

Statically Indeterminate Beams and Shafts

Compatibility equation

Thus Equation(1) becomes

$$0.012EI = 640 - 10.67B_y$$

Expressing *E* and *I* in units of kN/m^2 and m^4, we have

$$0.012(200)\left(10^6\right)\left[80\left(10^{-6}\right)\right] = 640 - 10.67B_y$$

$$B_y = 42.0 \text{ kN} \uparrow$$

Statically Indeterminate Beams and Shafts

Equilibrium equations:

Applying this result to the beam, we then calculate the reactions at A and C using eqns of equilibrium.

$$\curvearrowleft + \sum M_A = 0; \quad -96\text{ kN}(2\text{ m}) + 42.0\text{ kN}(4\text{ m}) + C_y(8\text{ m}) = 0$$

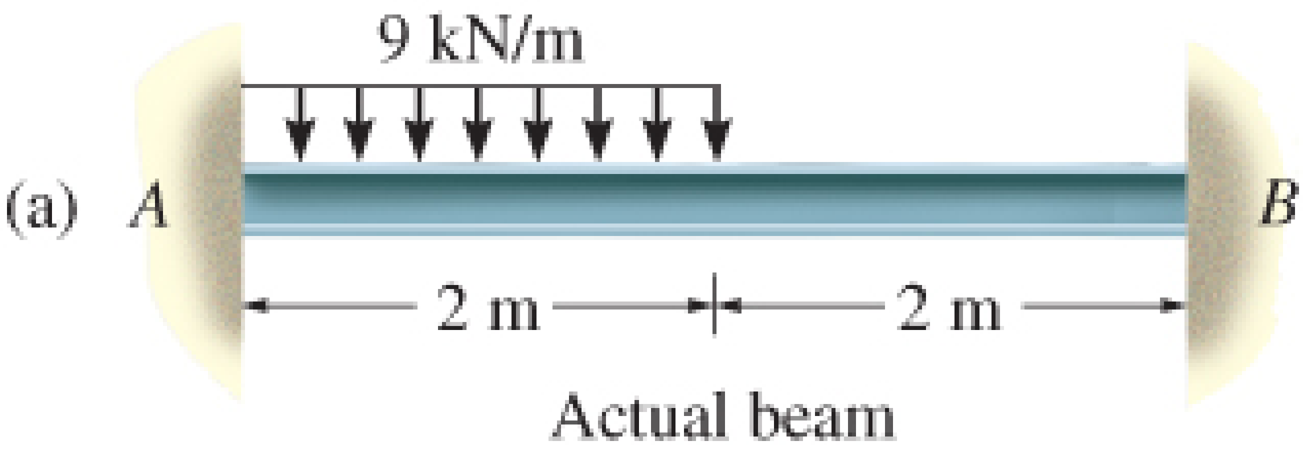
$$C_y = 3.00\text{ kN} \uparrow$$

$$+\uparrow \sum F_y = 0; \quad Ay - 96\text{ kN} + 42.0\text{ kN} + 3.00\text{ kN} = 0$$

$$A_y = 51\text{ kN} \uparrow$$

Statically Indeterminate Beams and Shafts

Determine the moment at *B* for beam shown. *EI* is constant. Neglect the effects of axial load.

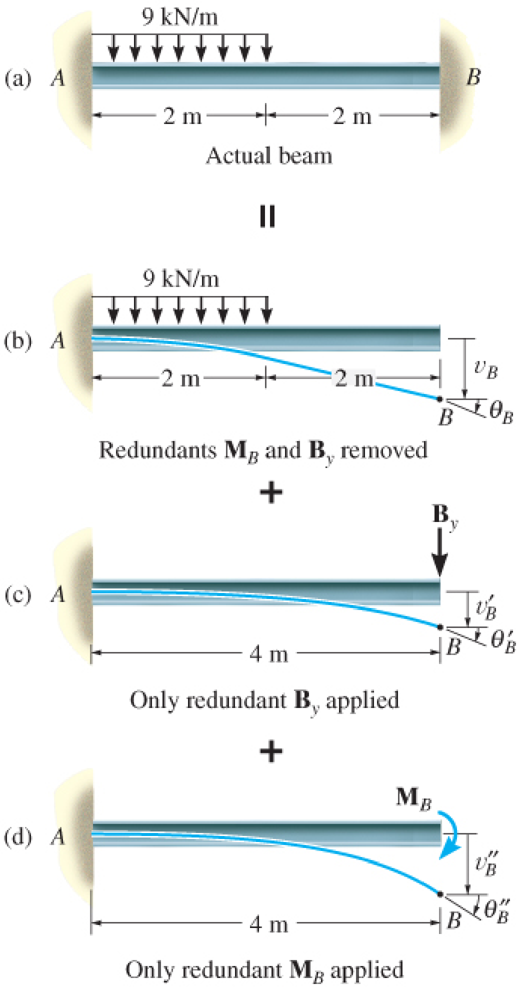


Statically Indeterminate Beams and Shafts

Principle of superposition:

Since axial load if neglected, a there is a vertical force and moment at *A* and *B*. Since only two equations of equilibrium are available, problem is indeterminate to the second degree.

Assume that $\mathbf{B}_y$ and $\mathbf{M}_B$ are redundant, so that by principle of superposition, beam is represented as a cantilever, loaded separately by distributed load and reactions $\mathbf{B}_y$ and $\mathbf{M}_B$, as shown.



Statically Indeterminate Beams and Shafts

Compatibility equations:

Referring to displacement and slope at *B*, we require

$$(\circlearrowleft +) \qquad 0 = \theta_B + \theta'_B + \theta''_B \qquad (1)$$

$$(+\downarrow) \qquad 0 = \upsilon_B + \upsilon'_B + \upsilon''_B \qquad (2)$$

Using table in Appendix C to compute slopes and displacements, we have

$$\theta_B = \frac{wL^3}{48EI} = \frac{9\ \text{kN/m}(4\ \text{m})^3}{48EI} = \frac{12}{EI} \circlearrowright$$

$$\upsilon_B = \frac{7wL^4}{384EI} = \frac{7(9\ \text{kN/m})(4\ \text{m})^4}{384EI} = \frac{42}{EI} \downarrow$$

Statically Indeterminate Beams and Shafts

Compatibility equations:

$$\theta'_B = \frac{PL^2}{2EI} = \frac{B_y(4\text{ m})^2}{2EI} = \frac{8B_y}{EI} \curvearrowright$$

$$\upsilon'_B = \frac{PL^3}{3EI} = \frac{B_y(4\text{ m})^3}{3EI} = \frac{21.33B_y}{EI} \downarrow$$

$$\theta''_B = \frac{ML}{EI} = \frac{M_B(4\text{ m})}{EI} = \frac{4M_B}{EI} \curvearrowright$$

$$\upsilon''_B = \frac{ML^2}{2EI} = \frac{M_B(4\text{ m})^2}{2EI} = \frac{8M_B}{EI} \downarrow$$

Statically Indeterminate Beams and Shafts

Compatibility equations:

Substituting these values into Eqns (1) and (2) and canceling out the common factor *EI*, we have

$$(\circlearrowleft +) \qquad 0 = 12 + 8B_y + 4M_B$$

$$(+\downarrow) \qquad 0 = 42 + 21.33B_y + 8M_B$$

Solving simultaneously, we get

$$B_y = -3.375 \text{ kN}$$

$$M_B = 3.75 \text{ kN} \cdot \text{m}$$