Lecture 10

Deflections of Beams and Shafts (1)

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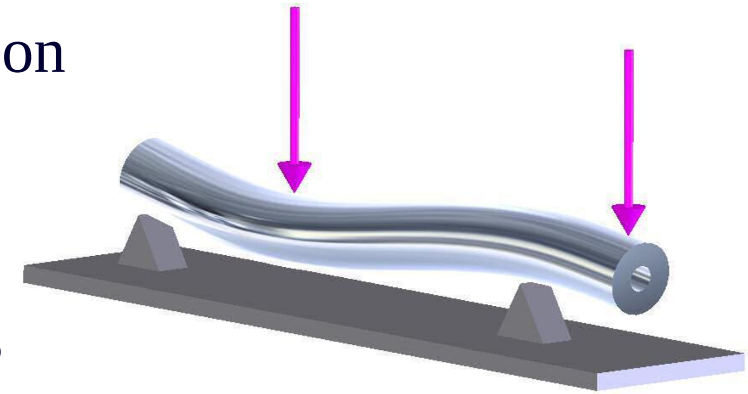
Lecture Objectives

- ✓ Use various methods to determine the deflection and slope at specific points on beams and shafts:
  1. Integration method
  2. Discontinuity functions
  3. Method of superposition
  4. Moment-area method



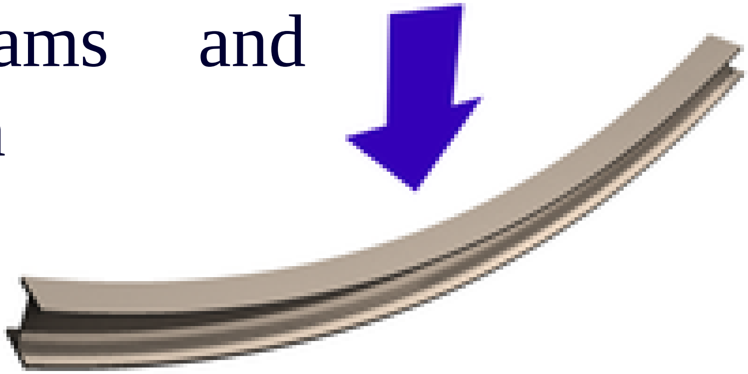
- ✓ Use the various methods to solve for the support reactions on a beam or shaft that is statically indeterminate

Lecture Outline

- ✓ The Elastic Curve
- ✓ Slope and Displacement by Integration
- ✓ Discontinuity Functions
- ✓ Slope and Displacement by the Moment-Area Method
- ✓ Method of Superposition
- ✓ Statically Indeterminate Beams and Shafts
- ✓ Statically Indeterminate Beams and Shafts: Method of Integration

Lecture Outline

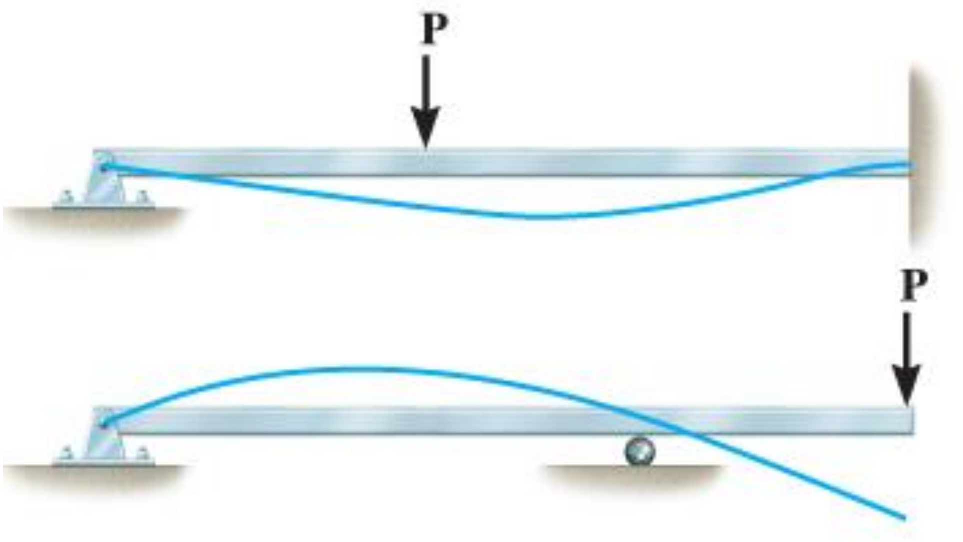
- ✓ Statically Indeterminate Beams and Shafts: Moment-Area Method
- ✓ Statically Indeterminate Beams and Shafts: Method of Superposition



The Elastic Curve

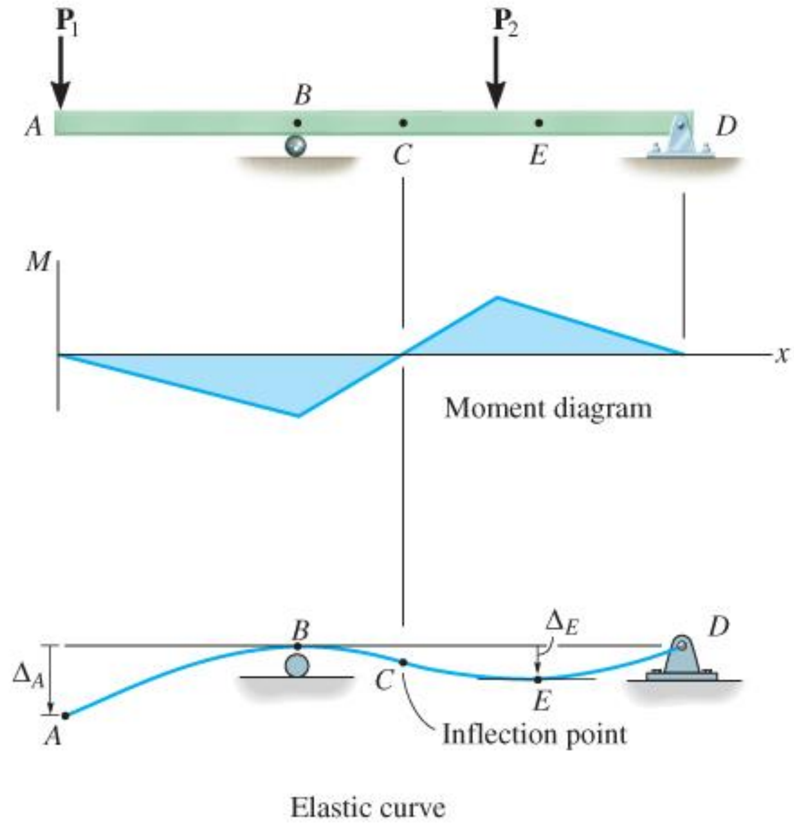
- It is useful to sketch the deflected shape of the loaded beam, to "visualize" computed results and partially check the results.

- The deflection diagram of the longitudinal axis that passes through the centroid of each x-sectional area of the beam is called the elastic curve.



The Elastic Curve

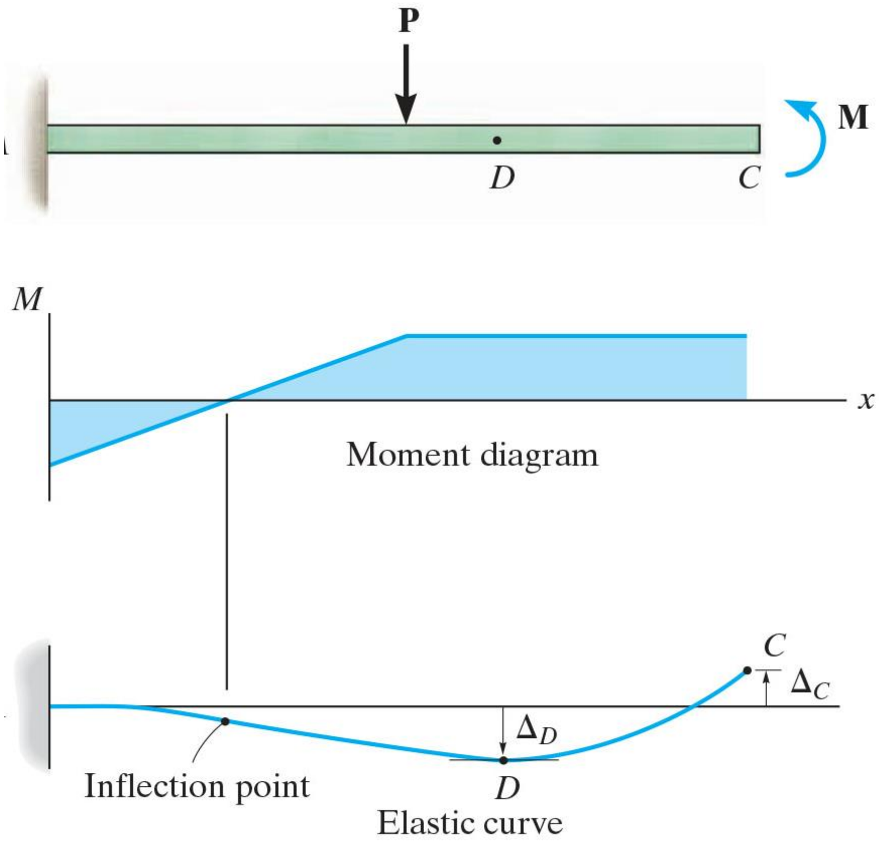
- Roller support at $B \Rightarrow$ displacements is zero
- Pin supports at $D \Rightarrow$ displacements is zero
- $A \rightarrow C$: *negative* moment $\Rightarrow$ elastic curve concave downwards
- $C \rightarrow D$: *positive* moment $\Rightarrow$ elastic curve concave upwards



- At C, there is an inflection point where curve changes from concave up to concave down (zero moment).

The Elastic Curve

- Fixed support at A $\Rightarrow$ displacements & slop are zero
- $A \rightarrow E$: *negative* moment $\Rightarrow$ elastic curve concave downwards
- $E \rightarrow C$: *positive* moment $\Rightarrow$ elastic curve concave upwards



- At E, there is an inflection point where curve changes from concave up to concave down (zero moment).

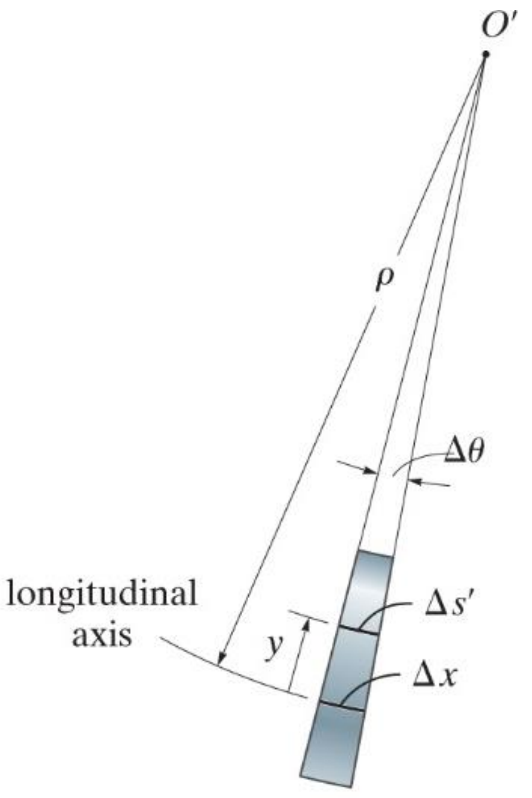
Moment-Curvature Relationship

- It's found that

$$\varepsilon = -\frac{y}{\rho}$$

The Curvature ($1/\rho$)

$$\frac{1}{\rho} = -\frac{\varepsilon}{y}$$



Deformed element

Moment-Curvature Relationship

If the material is homogeneous and shows linear-elastic behavior, Hooke's law applies. Since flexure formula also applies, we combing the equations to get

$$\left.\begin{aligned} 1/\rho &= -\varepsilon/y \\ \sigma &= \varepsilon.E \\ \sigma &= -M.y/I \end{aligned}\right\} \frac{1}{\rho} = \frac{M}{EI}$$

ρ : radius of curvature at a specific point on elastic curve

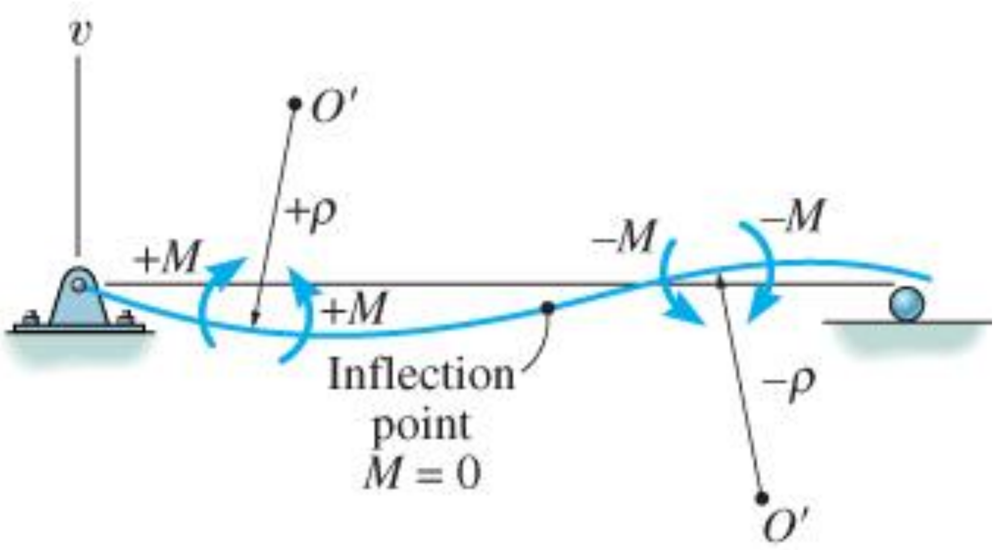
M: internal moment in beam at point where is to be determined.

E : material's modulus of elasticity.

I : beam's moment of inertia computed about neutral axis.

Moment-Curvature Relationship

- EI is the flexural rigidity and is always positive.
- Sign for ρ depends on the direction of the moment.
  - ✓ when M is positive, ρ extends above the beam.
  - ✓ When M is negative, ρ extends below the beam.



Stress-Curvature Relationship

Using flexure formula, curvature is also

$$\left.\begin{aligned} \frac{1}{\rho} &= \frac{M}{EI} \\ \sigma &= -\frac{M\,.y}{I} \end{aligned}\right\} \frac{1}{\rho} = -\frac{\sigma}{Ey}$$

Moment and Stress-Curvature Relationships are valid for either small or large radii of curvature

Slope and Displacement

1. Integration method
2. Discontinuity functions
3. Method of superposition
4. Moment-area method

Slope and displacement by integration

The equation of the elastic curve for a beam can be expressed mathematically as:

$$\upsilon = f\,(x)$$

Let's represent the curvature in terms of v and x.

$$\frac{1}{\rho} = \frac{d^2\upsilon/dx^2}{\left[1+\left(d\upsilon/dx\right)^2\right]^{3/2}}$$

$$\frac{1}{\rho} = \frac{M}{EI} \Rightarrow \frac{d^2\upsilon/dx^2}{\left[1+\left(d\upsilon/dx\right)^2\right]^{3/2}} = \frac{M}{EI}$$

Slope and displacement by integration

Most engineering codes specify limitations on deflections for tolerance or aesthetic purposes.

Slope of elastic curve determined from *dv/dx* is very small and its square will be negligible compared with unity.

Therefore, by approximation

$$\frac{1}{\rho}=\frac{M}{EI}\Rightarrow\frac{d^2\upsilon}{dx^2}=\frac{M}{EI}\Rightarrow M=EI\frac{d^2\upsilon}{dx^2}$$

Differentiate each side with respect to x

$$V(x)=\frac{dM}{dx}=\frac{d}{dx}\left(EI\frac{d^2\upsilon}{dx^2}\right)=EI\frac{d^3\upsilon}{dx^3}$$ Flexural rigidity is constant along beam

Slope and displacement by integration

Differentiating again with respect to x we get

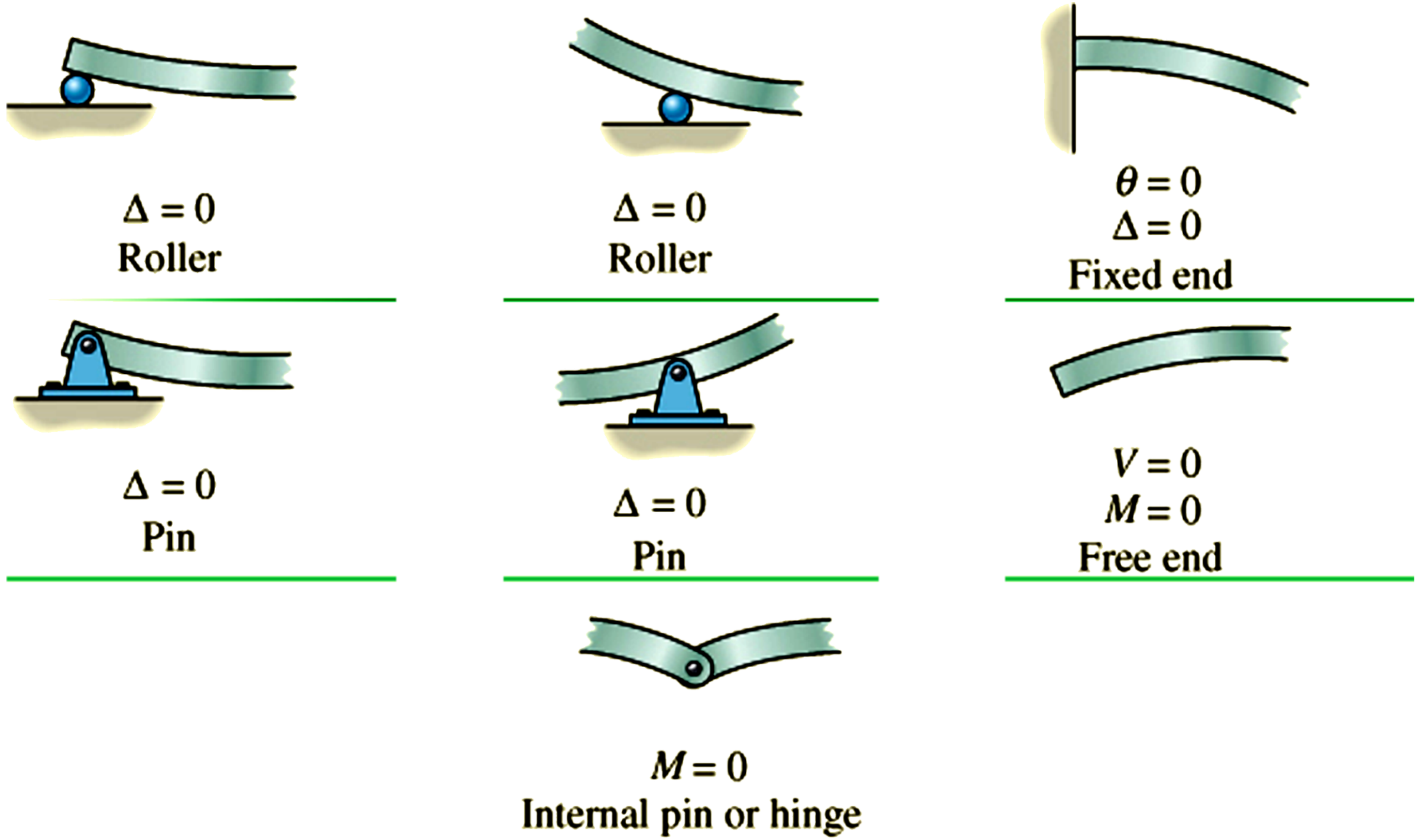
$$w(x) = \frac{dV}{dx} = \frac{d^2}{dx^2}\left(EI\frac{d^2\upsilon}{dx^2}\right) = EI\frac{d^4\upsilon}{dx^4}$$ Flexural rigidity is constant along beam

Generally, it is easier to determine the internal moment M as a function of x, integrate twice, and evaluate only two integration constants.

For convenience in writing each moment expression, the origin for each x coordinate can be selected arbitrarily.

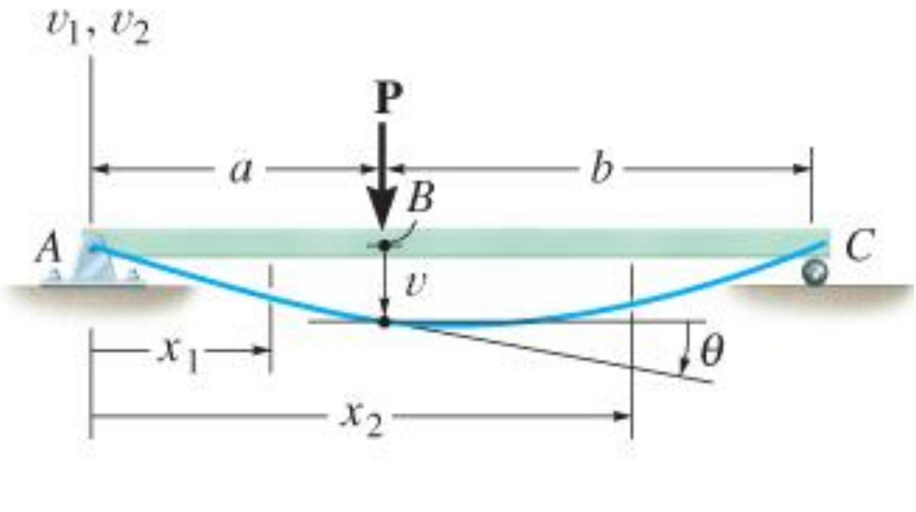
Slope and displacement by integration

Possible boundary conditions are:



Slope and displacement by integration

If a single x coordinate cannot be used to express the equation for beam's slope or elastic curve, then continuity conditions must be used to evaluate some of the integration constants.



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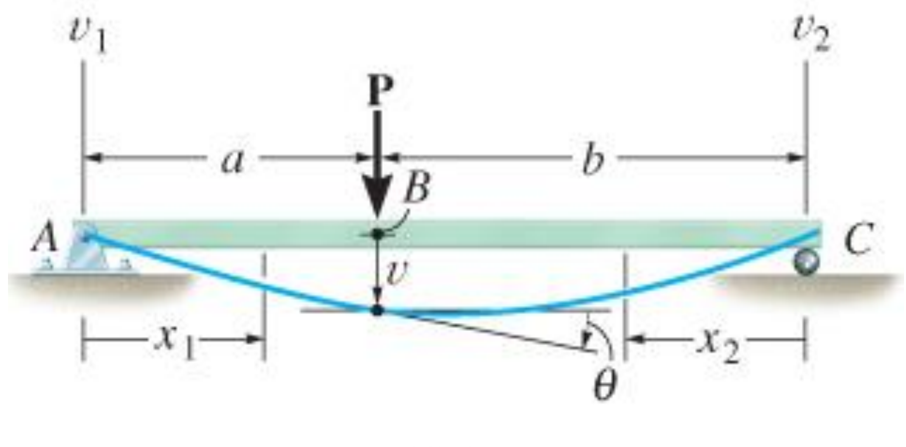


(b)

Slope and displacement by integration

Elastic Curve.

- Draw an exaggerated view of the beam's elastic curve. Recall that zero slope and zero displacement occur at all fixed supports, and zero displacement occurs at all pin and roller supports.
- Establish the x and v coordinate axes. The x axis must be parallel to the undeflected beam and can have an origin at any point along the beam, with a positive direction either to the right or to the left.
- If several discontinuous loads are present, establish x coordinates that are valid for each region of the beam between the discontinuities. Choose these coordinates so that they will simplify subsequent algebraic work.
- In all cases, the associated positive v axis should be directed upward.

Slope and displacement by integration

Load or Moment Function.

- For each region in which there is an x coordinate, express the loading w or the internal moment M as a function of x. In particular, *always* assume that M acts in the *positive direction* when applying the equation of moment equilibrium to determine $M = f(x)$.

Slope and displacement by integration

Slope and Elastic Curve.

- Provided EI is constant, apply either the load equation $EI\,d^4v/dx^4 = w(x)$, which requires four integrations to get $v = v(x)$, or the moment equation $EI\,d^2v/dx^2 = M(x)$, which requires only two integrations. For each integration it is important to include a constant of integration.
- The constants are evaluated using the boundary conditions for the supports (Table 12–1) and the continuity conditions that apply to slope and displacement at points where two functions meet. Once the constants are evaluated and substituted back into the slope and deflection equations, the slope and displacement at *specific points* on the elastic curve can then be determined.
- The numerical values obtained can be checked graphically by comparing them with the sketch of the elastic curve. Realize that *positive* values for *slope* are *counterclockwise* if the x axis extends *positive* to the *right*, and *clockwise* if the x axis extends *positive* to the *left*. In either of these cases, *positive displacement* is *upward*.

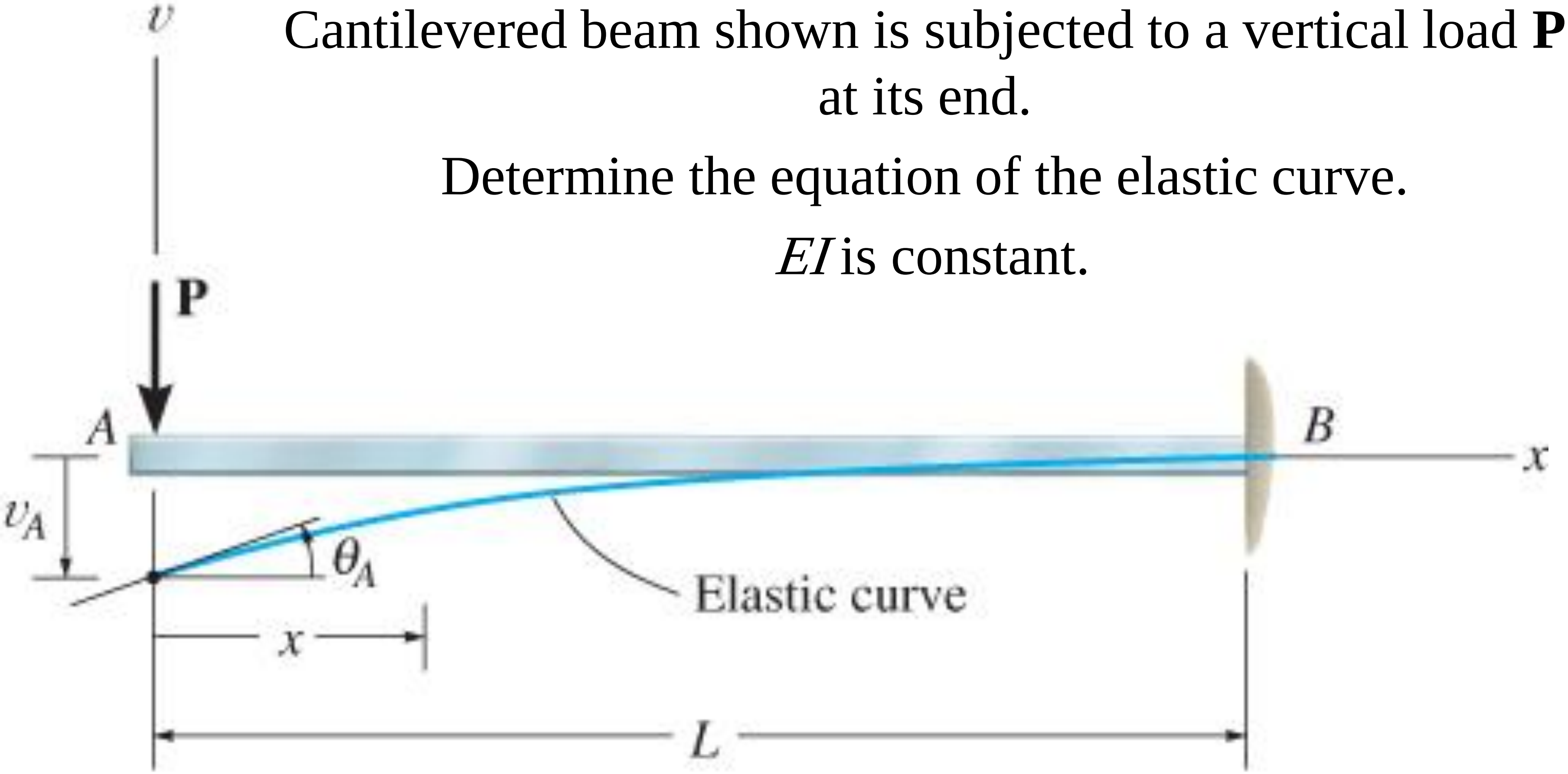
Slope and displacement by integration

Example:

Cantilevered beam shown is subjected to a vertical load **P** at its end.

Determine the equation of the elastic curve.

EI is constant.



Slope and displacement by integration

Example:

Elastic curve:

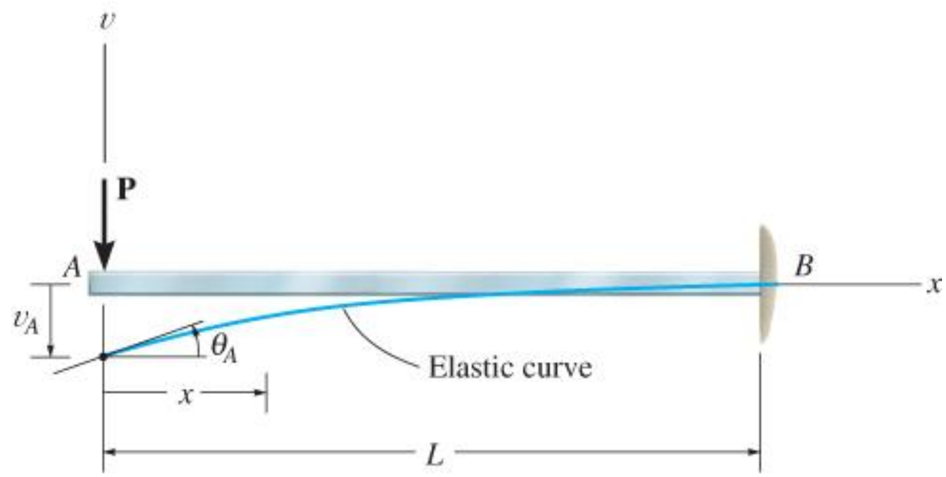
Load tends to deflect the beam.

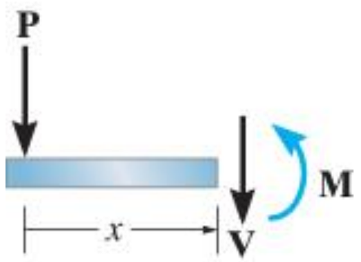
By inspection, the internal moment can be represented throughout the beam using a single x coordinate.

Moment function:

From free-body diagram, with **M** acting in the positive direction, we have

$$M = -Px$$





(b)

Slope and displacement by integration

Example:

Slope and elastic curve:

$$EI\frac{d^2\upsilon}{dx^2} = -Px \qquad (1)$$

$$EI\frac{d\upsilon}{dx} = -\frac{Px^2}{2} + C_1 \qquad (2)$$

$$EI\upsilon = -\frac{Px^3}{6} + C_1x + C_2 \qquad (3)$$

Slope and displacement by integration

Example:

Slope and elastic curve:

Using boundary conditions

$$\left.\begin{array}{lll} d\upsilon/dx = 0 & \text{at} & x = l \\ \upsilon = 0 & \text{at} & x = l \end{array}\right\} \Rightarrow \begin{cases} 0 = -\dfrac{PL^2}{2} + C_1 & \Rightarrow C_1 = \dfrac{PL^2}{2} \\ 0 = -\dfrac{PL^3}{6} + C_1 L + C_2 & \Rightarrow C_2 = -\dfrac{PL^3}{3} \end{cases}$$

Slope and displacement by integration

Example:

Slope and elastic curve:

$$EI\frac{d\upsilon}{dx} = -\frac{Px^2}{2} + \frac{PL^2}{2} \Rightarrow \theta = \frac{P}{2EI}(L^2 - x^2)$$

$$EI\upsilon = -\frac{Px^3}{6} + \frac{PL^2}{2}x + -\frac{PL^3}{3} \Rightarrow \upsilon = \frac{P}{6EI}(-x^3 + 3L^2x - 2L^3)$$

Maximum slope and displacement occur at A $(x = 0)$,

$$\theta_A = \frac{PL^2}{2EI} \qquad \upsilon_A = -\frac{PL^3}{3EI}$$

Slope and displacement by integration

Example:

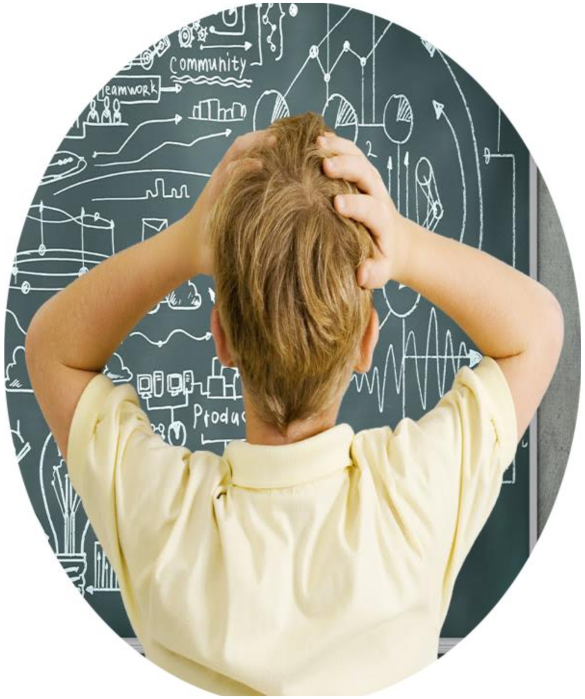
Slope and elastic curve:

Positive result for θ_A indicates counterclockwise rotation and negative result for v_A indicates that v_A is downward.

Consider beam to have a length of 5 m, support load P = 30 kN and made of A-36 steel having E_{st} = 200 GPa.

Slope and displacement by discontinuity functions

The method of integration, used to find the equation of the elastic curve for a beam or shaft, is convenient if the load or internal moment can be expressed as a continuous function throughout the beam's entire length. If several different loadings act on the beam, however, the method becomes more tedious to apply, because separate loading or moment functions must be written for each region of the beam. Furthermore, integration of these functions requires the evaluation of integration constants using both boundary and continuity conditions.



Slope and displacement by discontinuity functions

Simplified Method

Equation of the elastic curve for a multiply loaded beam using a single expression from

the loading on the beam

$$w = w(x)$$

the beam's internal moment

$$M = M(x)$$

Slope and displacement by discontinuity functions

Two types of mathematical operators known as discontinuity functions

1. Macaulay Functions

2. Singularity Functions

Slope and displacement by discontinuity functions

1. Macaulay Functions

$$\langle x-a\rangle^{n}=\begin{cases}0 & for\ x<a\\ (x-a)^{n} & for\ x\geq a\end{cases}$$

$$n\geq 0$$

Slope and displacement by discontinuity functions

- *x* represents the coordinate position of a point along the beam
- *a* is the location on the beam where a "discontinuity" occurs, or the point where a distributed loading begins.
- Integrating Macaulay functions, we get

$$\int \langle x - a \rangle^{n} \, dx = \frac{\langle x - a \rangle^{n+1}}{n+1} + C$$

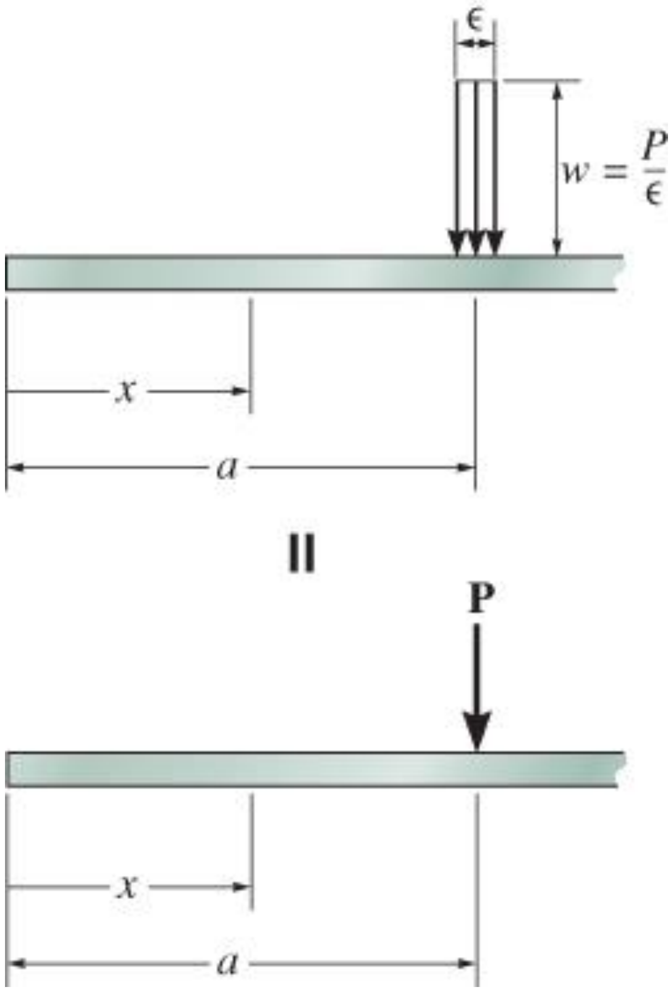
- The functions describe both uniform load and triangular load.

Slope and displacement by discontinuity functions

2. Singularity functions

A concentrated force P can be considered as a special case of distributed loading, where $w = P/e$ such that its width is ε, $\varepsilon \rightarrow 0$.

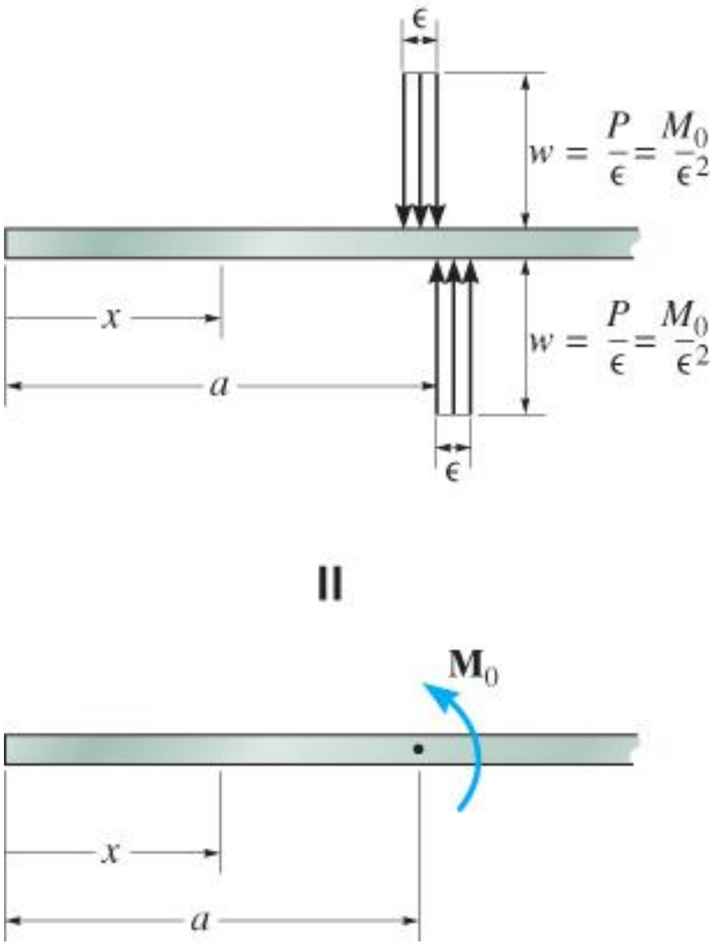
$$w = P\langle x - a\rangle^{-1} = \begin{cases} 0 & for\ x \neq a \\ P & for\ x = a \end{cases}$$



Slope and displacement by discontinuity functions

2. Singularity functions

couple moment M0, considered positive counterclockwise, is a limitation as $\varepsilon \to 0$. of two distributed loadings.



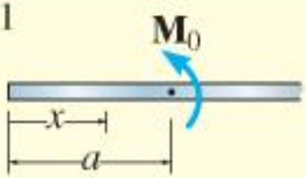
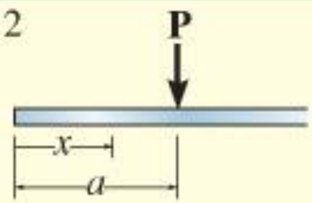
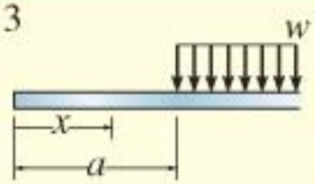
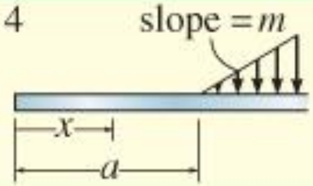
$$w = M_0 \langle x - a \rangle^{-2} = \begin{cases} 0 & for\ x \neq a \\ M_0 & for\ x = a \end{cases}$$

Slope and displacement by discontinuity functions

Integration of the two functions yields

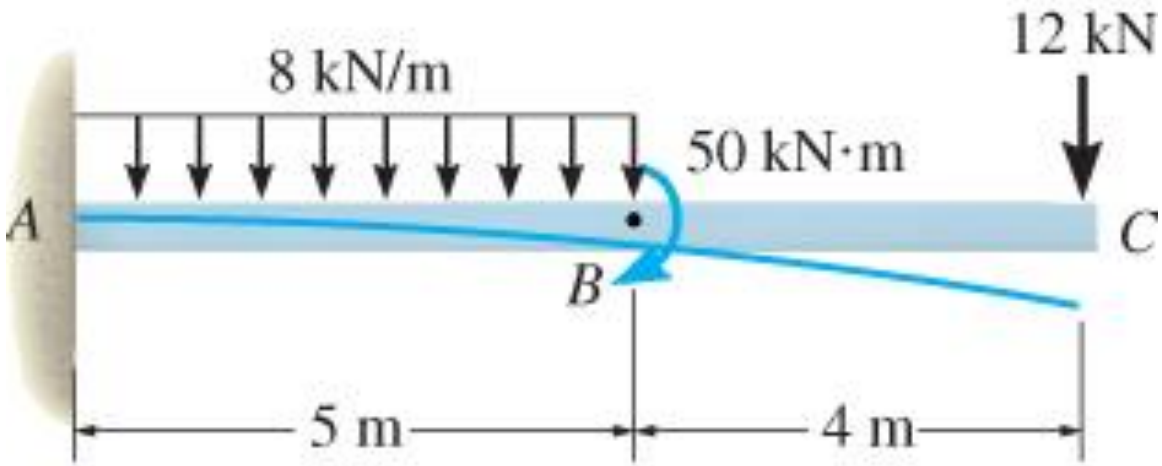
$$\int \langle x - a \rangle^{n} \, dx = \langle x - a \rangle^{n+1}, \; n = -1, -2$$

Slope and displacement by discontinuity functions

Loading	Loading Function $w = w(x)$	Shear $V = -\int w(x)dx$	Moment $M = \int V dx$
1 	$w = M_0 \langle x-a \rangle^{-2}$	$V = -M_0 \langle x-a \rangle^{-1}$	$M = -M_0 \langle x-a \rangle^{0}$
2 	$w = P \langle x-a \rangle^{-1}$	$V = -P \langle x-a \rangle^{0}$	$M = -P \langle x-a \rangle^{1}$
3 	$w = w_0 \langle x-a \rangle^{0}$	$V = -w_0 \langle x-a \rangle^{1}$	$M = -\frac{w_0}{2} \langle x-a \rangle^{2}$
4 	$w = m \langle x-a \rangle^{1}$	$V = \frac{-m}{2} \langle x-a \rangle^{2}$	$M = \frac{-m}{6} \langle x-a \rangle^{3}$

Slope and displacement by discontinuity functions

Determine the equation of the elastic curve for the cantilevered beam shown. *EI* is constant.

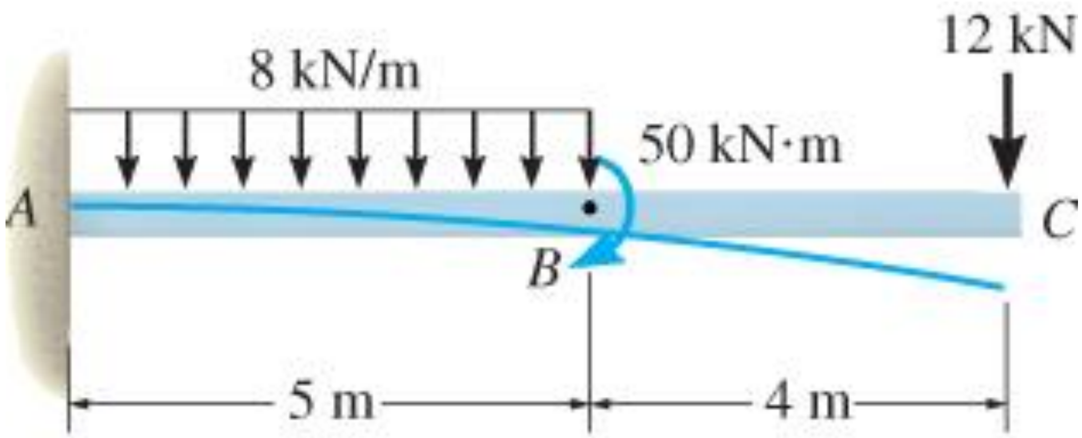


(a)

Slope and displacement by discontinuity functions

Elastic curve

The loads cause the beam to deflect as shown. The boundary conditions require zero slope and displacement at *A*.



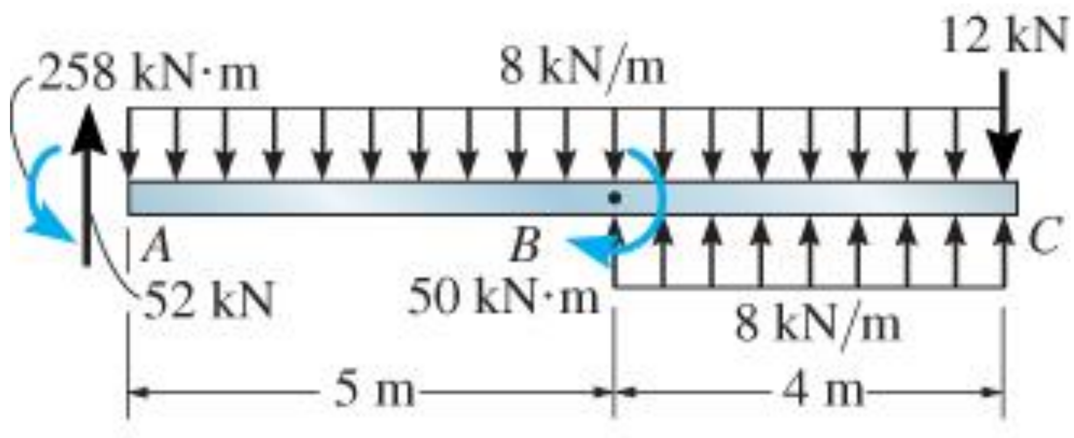
(a)

Slope and displacement by discontinuity functions

Loading functions

Support reactions shown on free-body diagram. Since distributed loading does not extend to *C* as required, use superposition of loadings to represent same effect.

By sign convention, the 50-kNm couple moment, the 52-kN force at *A*, and the distributed loading from *A* to *C* on the top of the beam are all negative.



(b)

Slope and displacement by discontinuity functions

Loading functions

Therefore,

$$w = 52\ \text{kN}\langle x-0\rangle^{-1} - 258\ \text{kN}\cdot\text{m}\langle x-0\rangle^{-2} - 8\ \text{kN/m}\langle x-0\,\text{m}\rangle^{0}$$
$$+50\ \text{kN}\cdot\text{m}\langle x-5\ \text{m}\rangle^{-2} + 8\ \text{kN/m}\langle x-5\ \text{m}\rangle^{0}$$

The 12-kN load is not included, since x cannot be greater than 9 m. Because $dV/dx = w(x)$, then by integrating, neglect constant of integration since reactions are included in load function, we have

$$v = 52\ \text{kN}\langle x-0\rangle^{0} - 258\ \text{kN}\cdot\text{m}\langle x-0\rangle^{-1} - 8\ \text{kN/m}\langle x-0\,\text{m}\rangle^{1}$$
$$+50\ \text{kN}\cdot\text{m}\langle x-5\ \text{m}\rangle^{-1} + 8\ \text{kN/m}\langle x-5\ \text{m}\rangle^{1}$$

Slope and displacement by discontinuity functions

Loading functions

Furthermore, *dM/dx = V*, so integrating again yields

$$M = 52\ \text{kN}\langle x-0\rangle^{1} - 258\ \text{kN}\cdot\text{m}\langle x-0\rangle^{0} - 8/2\ \text{kN/m}\langle x-0\,\text{m}\rangle^{2}$$
$$+50\ \text{kN}\cdot\text{m}\langle x-5\ \text{m}\rangle^{0} + 8/2\ \text{kN/m}\langle x-5\ \text{m}\rangle^{2}$$
$$\Rightarrow M = EI\frac{d^2\upsilon}{dx^2} = -258 + 52x - 4x^2 + 50\langle x-5\rangle^{0} + 4\langle x-5\rangle^{2}$$

The same result can be obtained directly from the Table.

Slope and displacement by discontinuity functions

Slope and elastic curve

integrating twice, we have

$$EI\frac{d^2\upsilon}{dx^2} = -258 + 52x - 4x^2 + 50\langle x-5\rangle^0 + 4\langle x-5\rangle^2$$

$$EI\frac{d\upsilon}{dx} = -258x + 26x^2 - \frac{4}{3}x^3 + 50\langle x-5\rangle^1 + \frac{4}{3}\langle x-5\rangle^3 + C_1$$

$$EI\,\upsilon = -129x^2 + \frac{26}{3}x^3 - \frac{1}{3}x^4 + 25\langle x-5\rangle^2 + \frac{1}{3}\langle x-5\rangle^4 + C_1x + C_2$$

Slope and displacement by discontinuity functions

Slope and elastic curve

Since $dv/dx = 0$ at $x = 0$, $C_1 = 0$; and $v = 0$ at $x = 0$, so $C_2 = 0$. Thus

$$\upsilon = \frac{1}{EI}\left(-129x^2 + \frac{26}{3}x^3 - \frac{1}{3}x^4 + 25\langle x-5\rangle^2 + \frac{1}{3}\langle x-5\rangle^4\right)$$