Mechanics of Materials

Lecture 9

Bending (2)

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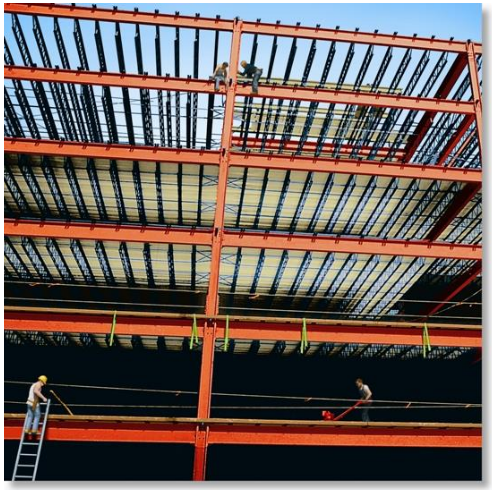
Lecture Objectives



- ✓ Determine stress in members caused by bending
- ✓ Discuss how to establish shear and moment diagrams for a beam or shaft
- ✓ Determine largest shear and moment in a member, and specify where they occur
- ✓ Consider members that are straight, symmetric x-section and homogeneous linear-elastic material
- ✓ Consider special cases of unsymmetrical bending and members made of composite materials

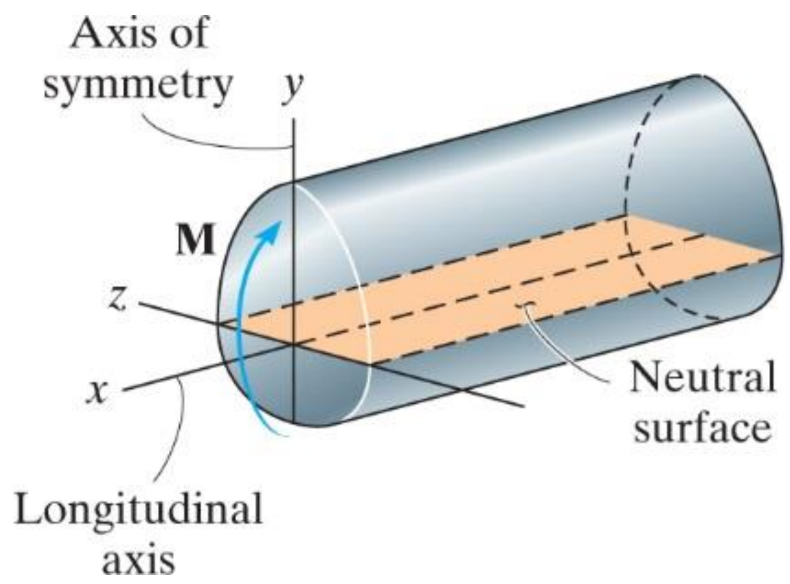
Lecture Outline

- ✓ Shear and Moment Diagrams
- ✓ Graphical Method for Constructing Shear and Moment Diagrams
- ✓ Bending Deformation of a Straight Member
- ✓ The Flexure Formula
- ✓ Unsymmetrical Bending

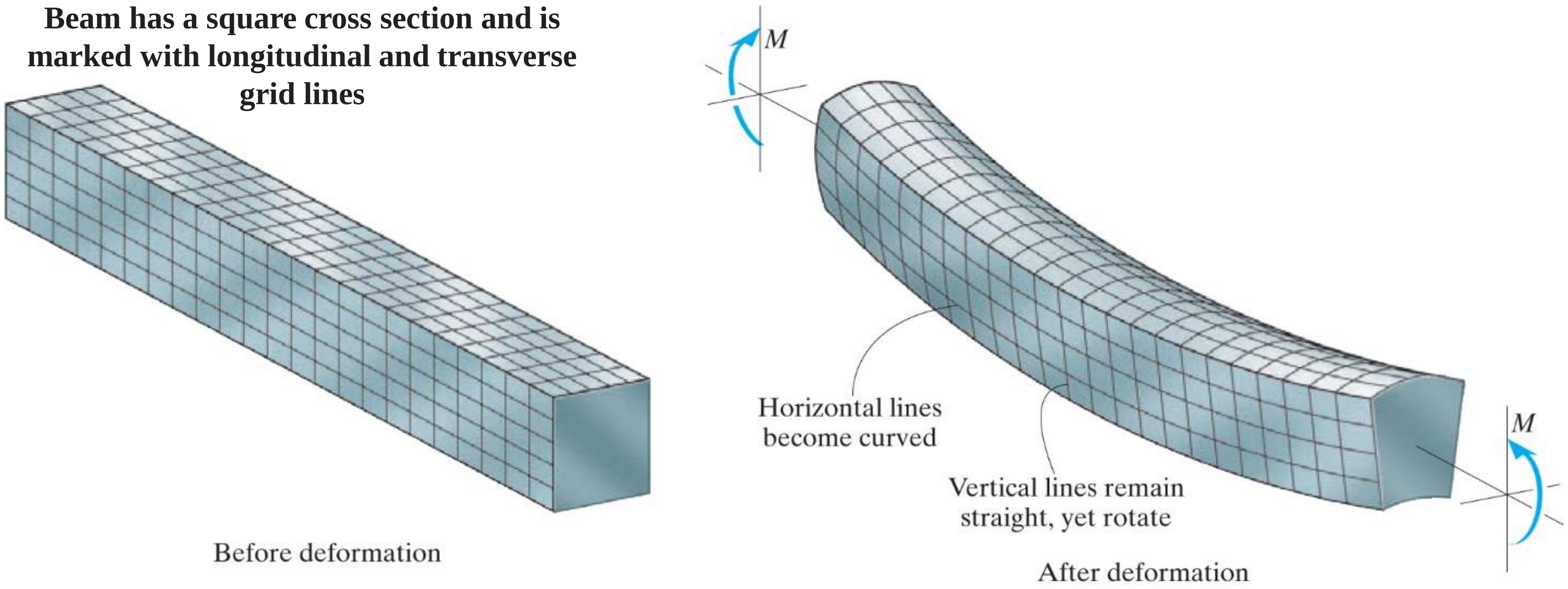
Bending Deformation of a Straight Member

Beam

- ✓ Straight prismatic
- ✓ Made of homogeneous material
- ✓ Subjected to bending
- ✓ Having a cross-sectional area that is symmetrical with respect to an axis
- ✓ Bending moment is applied about an axis perpendicular to this axis of symmetry



Bending Deformation of a Straight Member



The material within:

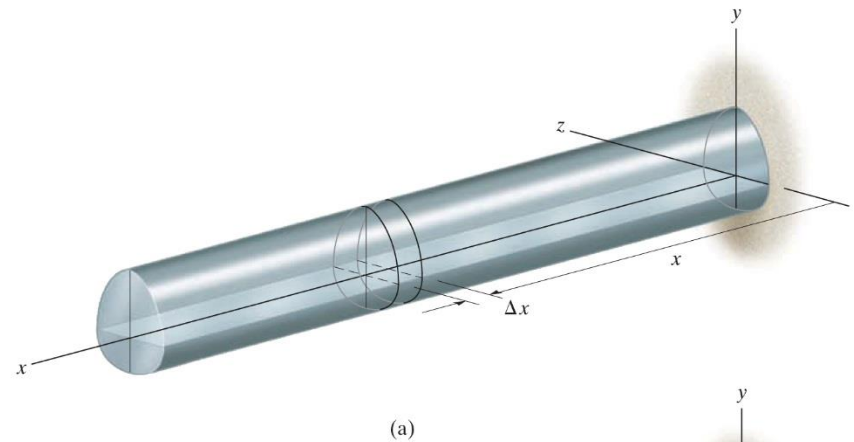
- Bottom portion of the bar ↔ stretch
- Top portion of the bar ↔ compress.

Between these two regions there must be a surface, called the neutral surface, in which longitudinal fibers of the material will not undergo a change in length

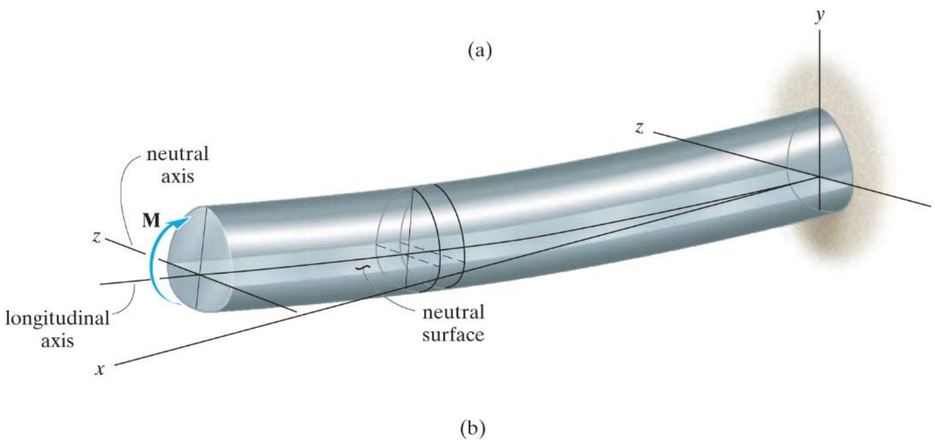
Bending Deformation of a Straight Member

- First, the *longitudinal axis x*, from the neutral surface, does *not* experience any *change in length.* It *becomes a curve* within *x–y* plane of symmetry,
- Second, all ***cross sections*** of the beam ***remain plane*** and perpendicular to the longitudinal axis during the deformation.
- Third, *any deformation* of the *cross section* within its own plane, will be *neglected.*

In particular, the *z* axis, lying in the plane of the cross section and about which the cross section rotates, is called the *neutral axis.*



(a)



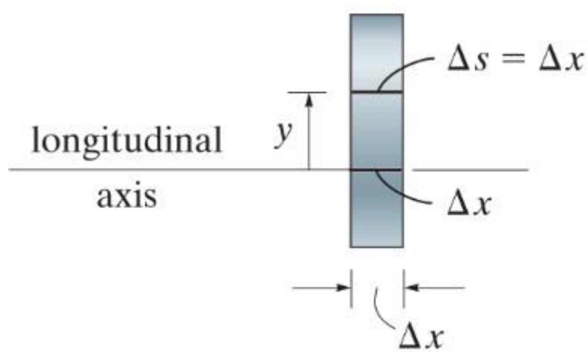
(b)

Bending Deformation of a Straight Member

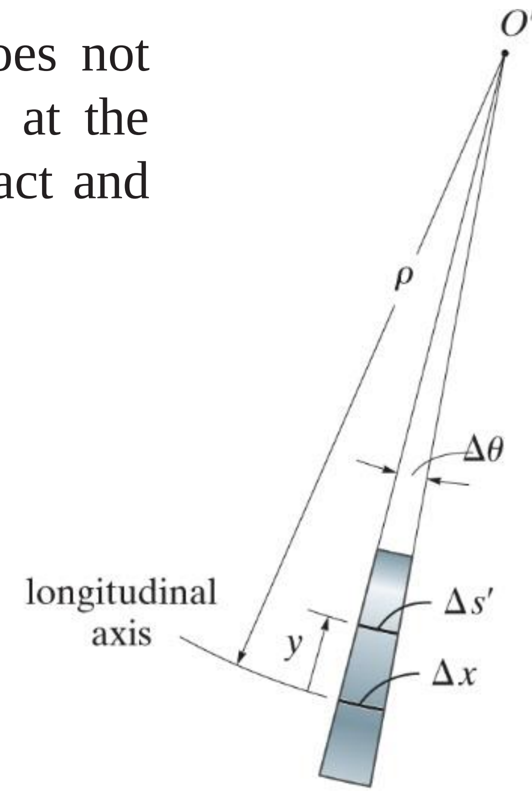
Any line segment Δx located on the neutral surface, does not change its length, whereas any line segment Δs located at the arbitrary distance *y* above the neutral surface, will contract and become after deformation Δs'.

By definition, the normal strain along Δs is determined as:

$$\varepsilon = \lim_{\Delta s \to 0} \frac{\Delta s' - \Delta s}{\Delta s}$$



Undeformed element



Deformed element

Bending Deformation of a Straight Member

Any line segment Δx located on the neutral surface, does not change its length, whereas any line segment Δs located at the arbitrary distance *y* above the neutral surface, will contract and become after deformation Δs'.

By definition, the normal strain along Δs is determined as,

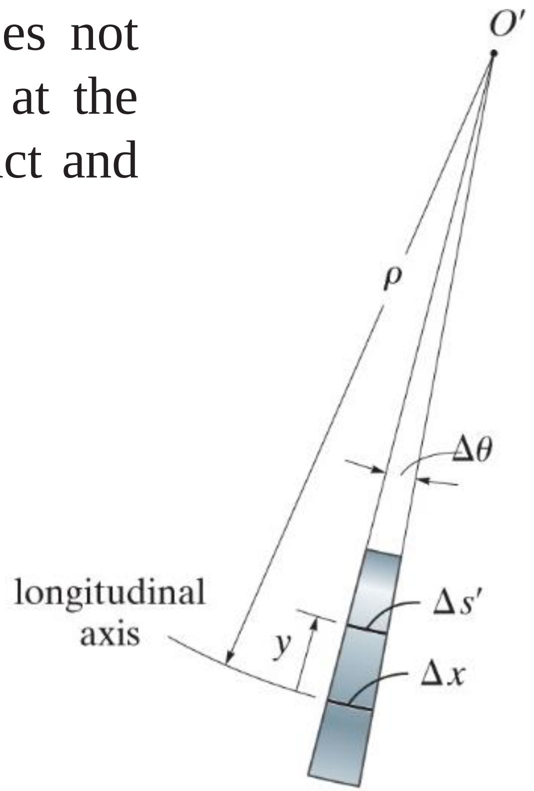
$$\varepsilon = \lim_{\Delta s \to 0} \frac{\Delta s' - \Delta s}{\Delta s}$$

$$\Delta s = \Delta x = \rho.\theta$$

$$\Delta s' = (\rho - y).\theta$$

$$\varepsilon = \lim_{\Delta \theta \to 0} \frac{(\rho - y).\theta - \rho.\theta}{\rho.\theta}$$

$$\varepsilon = -\frac{y}{\rho}$$



Deformed element

Bending Deformation of a Straight Member

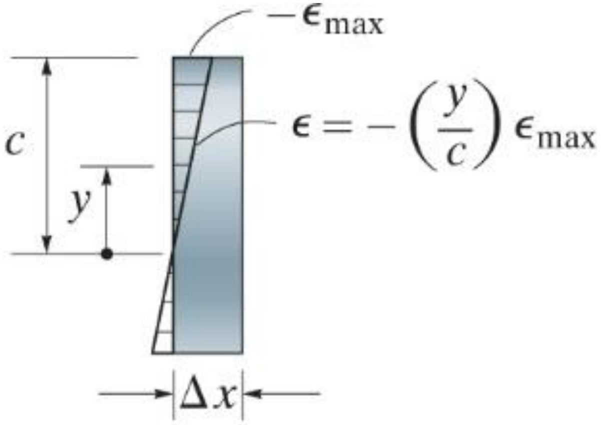
$$\varepsilon = -\frac{y}{\rho}$$

The longitudinal normal strain will vary linearly with y from the neutral axis.

A contraction -ε will occur in fibers located above the neutral axis +y whereas elongation +ε will occur in fibers located below the axis -y.

The maximum strain occurs at the outermost fiber, located a distance of y=c from the neutral axis.

$$\varepsilon_{\max} = \frac{c}{\rho}$$

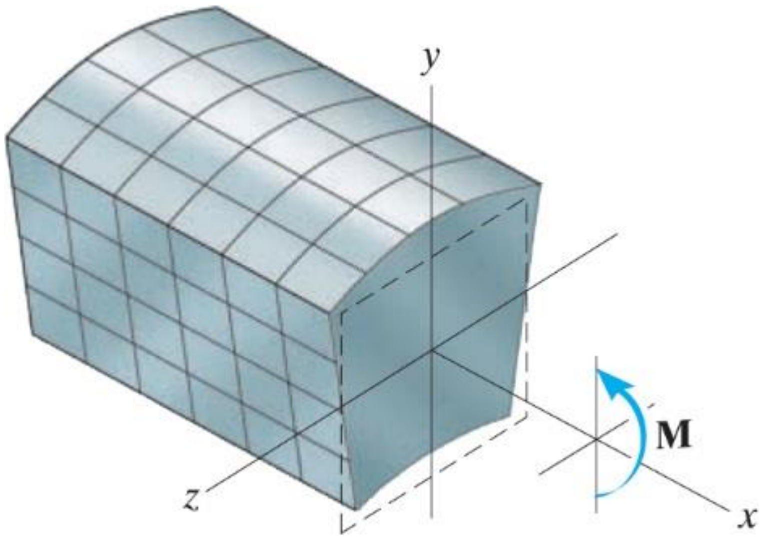


Normal strain distribution

$$\left.\begin{array}{l}\varepsilon = -y/\rho \\ \varepsilon_{\max} = c/\rho\end{array}\right\} \Rightarrow \varepsilon = -(y/c)\,\varepsilon_{\max}$$

Bending Deformation of a Straight Member

When a moment is applied to the beam, therefore, it will only cause a *normal stress* in the longitudinal or *x* direction. All the other components of normal and shear stress will be zero. It is this uniaxial state of stress that causes the material to have the longitudinal normal strain component. Furthermore, by Poisson's ratio, there must *also* be associated strain components which deform the plane of the cross-sectional area, although here we have neglected these deformations. Such deformations will, however, cause the *cross-sectional dimensions* to become smaller below the neutral axis and larger above the neutral axis.



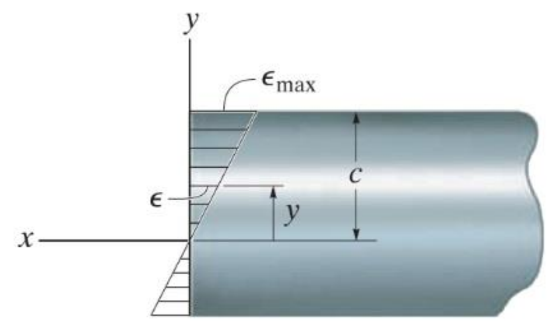
The Flexure Formula

Assume that:
Material behaves in a linear-elastic manner

$$\varepsilon = -\left(\frac{y}{c}\right)\varepsilon_{\max}$$

$$\sigma = E\,.\varepsilon$$

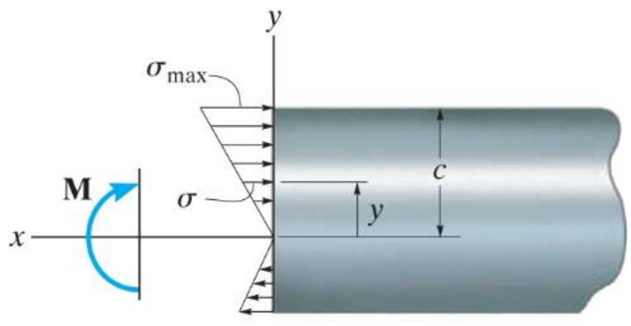
$$\sigma = -\left(\frac{y}{c}\right)\sigma_{\max}$$



Normal strain variation
(profile view)

(a)

For positive M which acts in the direction +y positive values of *y* give negative values for σ that is, a compressive stress since it acts in the negative *x* direction.
negative *y* values will give positive or tensile values for σ



Bending stress variation
(profile view)

The Flexure Formula

We can locate the position of the neutral axis on the cross section by satisfying the condition that the resultant force produced by the stress distribution over the cross-sectional area must be equal to zero.

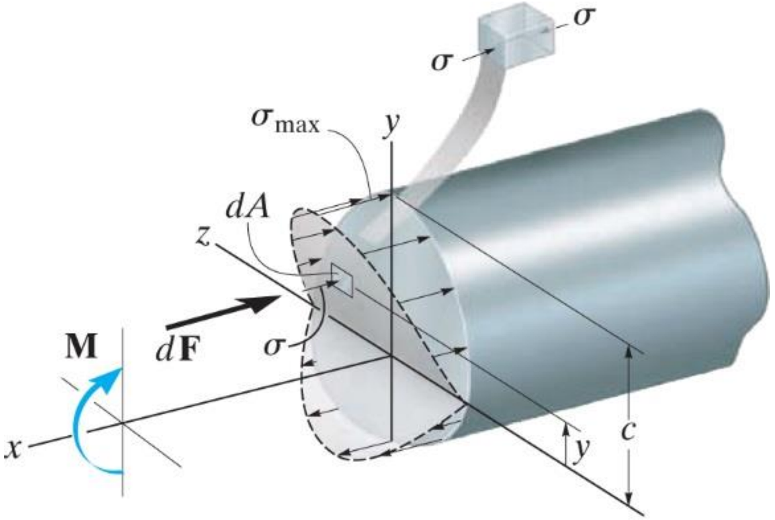
Noting that the force dF= σ.dA acts on the arbitrary element dA

$$F_R = \sum F_x ;$$

$$0 = \int_A \sigma . dA = \int_A -\left(\frac{y}{c}\right)\sigma_{\max} . dA$$

$$-\left(\frac{\sigma_{\max}}{c}\right)\int_A y . dA = 0$$

$$\int_A y . dA = 0$$



Bending stress variation

The Flexure Formula

We can determine the stress in the beam from the requirement that the resultant internal moment *M* must be equal to the moment produced by the stress distribution about the neutral axis.

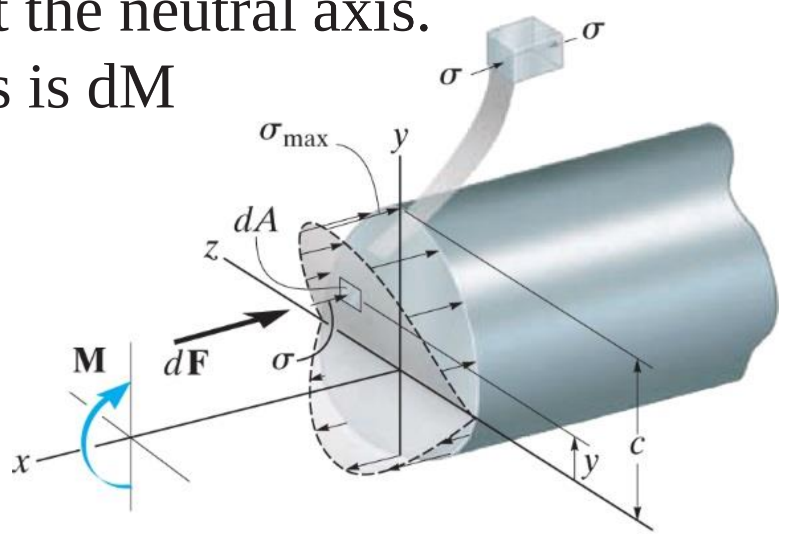
The moment of dF about the neutral axis is dM

$$(M_R)_z = \sum M_z ;$$

$$M = \int_A y.dF = \int_A y.\sigma.dA = \int_A y.\left(\frac{y}{c}\right)\sigma_{max}.dA$$

$$M = \left(\frac{\sigma_{max}}{c}\right)\int_A y^2.dA = \frac{\sigma_{max}}{c}.I$$

$$\Rightarrow \sigma_{max} = \frac{M.c}{I}$$



Bending stress variation

The integral represents the *moment of inertia* of the cross-sectional area about the neutral axis $I = \int_A y^2.dA$

The Flexure Formula

$$M = \left(\frac{\sigma_{max}}{c}\right)\int_A y^2 .dA = \frac{\sigma_{max}}{c}.I$$

$$\Rightarrow \sigma_{max} = \frac{M.c}{I}$$

σ_{max} The maximum normal stress in the member, which occurs at a point on the cross-sectional area farthest away from the neutral axis

M The resultant internal moment, determined from the method of sections and the equations of equilibrium, and calculated about the neutral axis of the cross section

c The perpendicular distance from the neutral axis to a point farthest away from the neutral axis. This is where acts

I The moment of inertia of the cross-sectional area about the neutral axis

The Flexure Formula

$$\left.\begin{aligned} \sigma_{\max} &= \frac{M\,.c}{I} \\ \sigma &= -\left(\frac{y}{c}\right)\sigma_{\max} \end{aligned}\right\} \sigma = -\frac{M\,.y}{I}$$

Geometric Parameters (Review)

Moments of Inertia (Rectangular CS)

$$I_z = \frac{a_{(z)} . b_{(y)}^{\ 3}}{12}$$

Moments of Inertia (Circular CS)

$$I_z = \frac{\pi}{4} r^4$$

Moments of Inertia (Combined CS)

$$I = \sum \bar{I} + A.\mathrm{d}$$

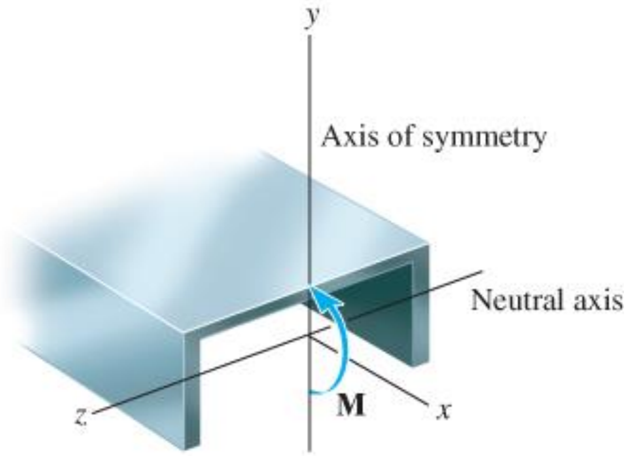
CS Centroid Coordinate

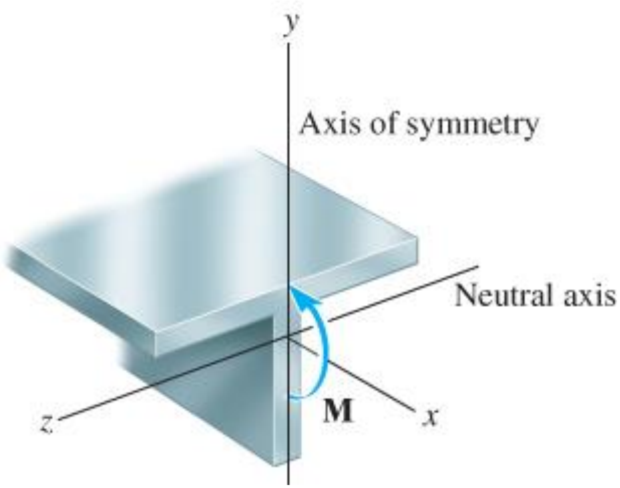
$$\bar{y} = \frac{\sum \bar{y}_i . A_i}{\sum A_i}$$

Unsymmetrical Bending

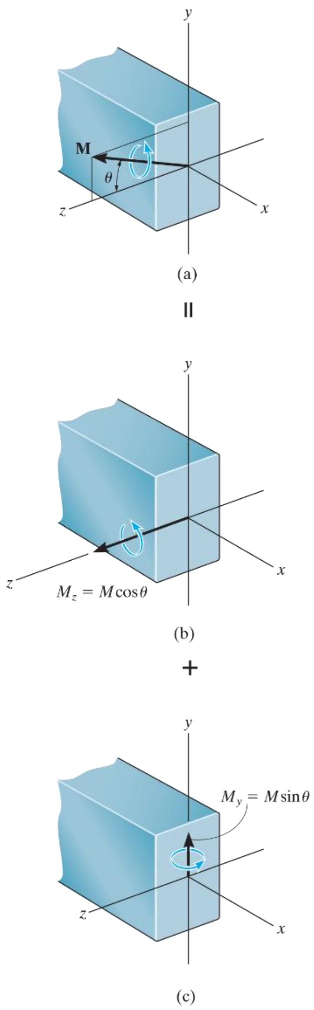
A condition for flexure formula is the symmetric x-sectional area of beam about an axis perpendicular to neutral axis

However, the flexure formula can also be applied either to a beam having x-sectional area of any shape OR to a beam having a resultant moment that acts in any direction





Unsymmetrical Bending



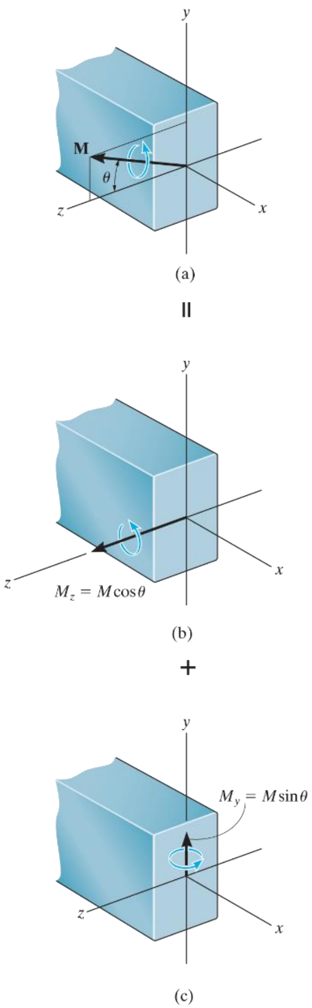
Moment Arbitrarily Applied

$$\sigma = -\frac{M_y . z}{I_y} + \frac{M_z . y}{I_z}$$

σ = normal stress at the point

y, z = coordinates of point measured from *x, y, z* axes having origin at centroid of x-sectional area and forming a right-handed coordinate system

Unsymmetrical Bending



Moment Arbitrarily Applied

$$\sigma = -\frac{M_y . z}{I_y} + \frac{M_z . y}{I_z}$$

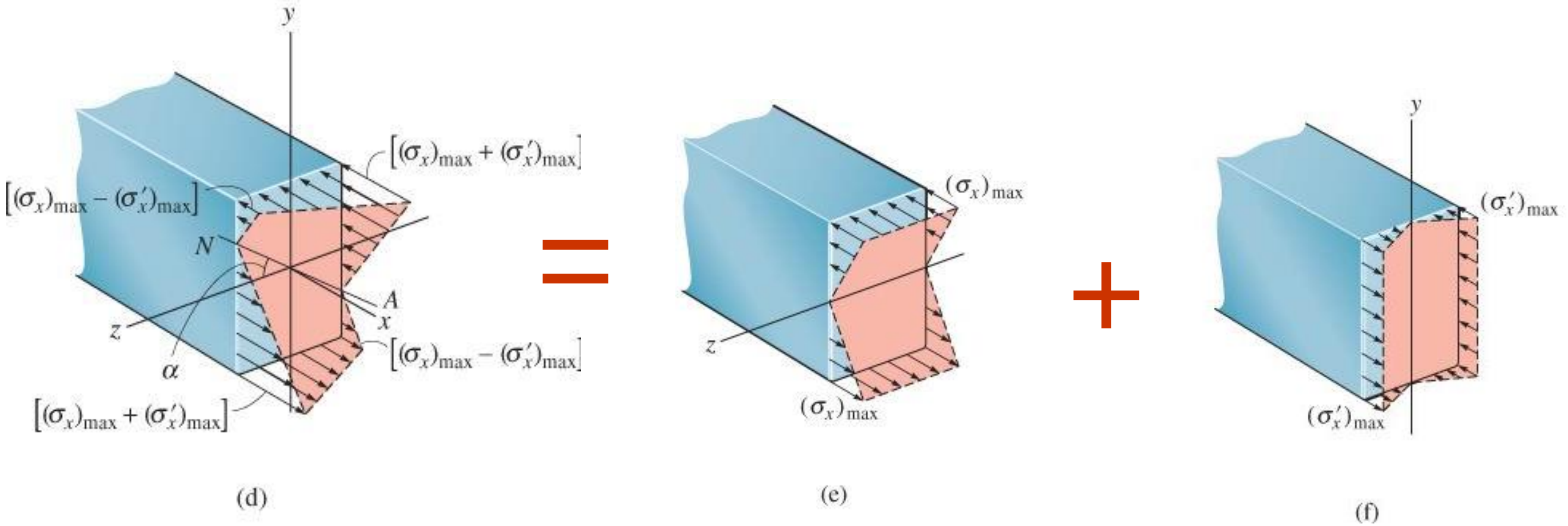
My, Mz = resultant internal moment components along principal y and z axes. Positive if directed along +y and +z axes. Can also be stated as My = M sin θ and Mz = M cos θ , where θ is measured positive from +z axis toward +y axis

Iy, Iz = principal moments of inertia computed about the y and z axes, respectively

Unsymmetrical Bending

Orientation of the Neutral Axis

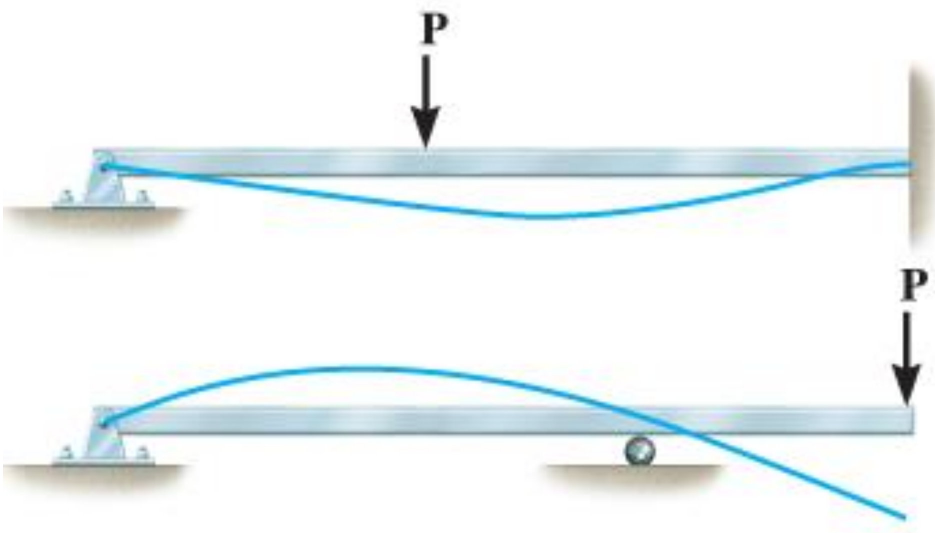
$$\tan\alpha = \frac{I_z}{I_y}\tan\theta$$



The Elastic Curve

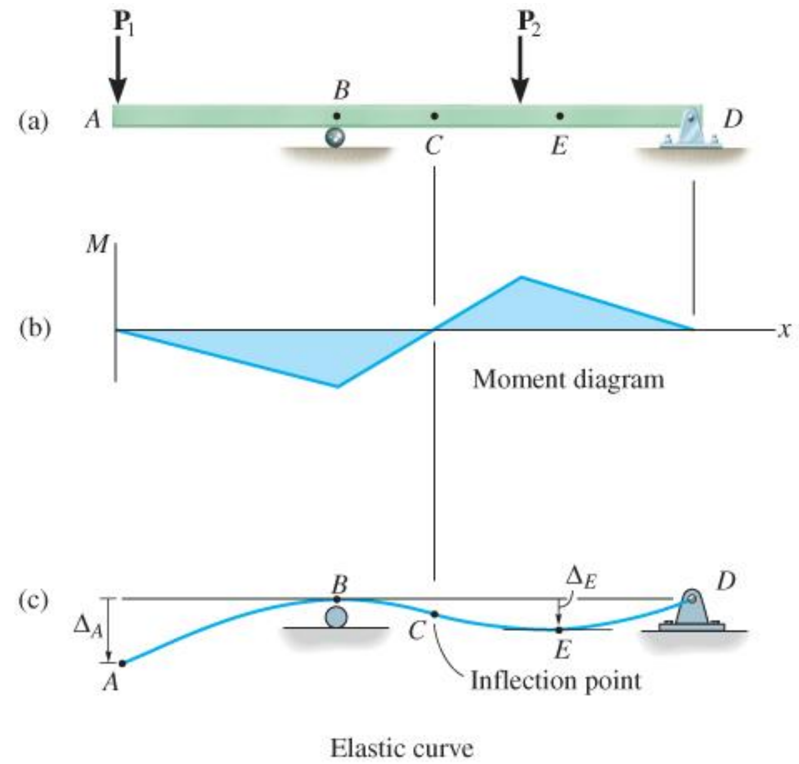
- It is useful to sketch the deflected shape of the loaded beam, to "visualize" computed results and partially check the results.

- The deflection diagram of the longitudinal axis that passes through the centroid of each x-sectional area of the beam is called the elastic curve.



The Elastic Curve

- Roller support at $B \Rightarrow$ displacements is zero
- Pin supports at $D \Rightarrow$ displacements is zero
- $A \to C$: *negative* moment $\Rightarrow$ elastic curve concave downwards
- $C \to D$: *positive* moment $\Rightarrow$ elastic curve concave upwards



- At C, there is an inflection point where curve changes from concave up to concave down (zero moment).

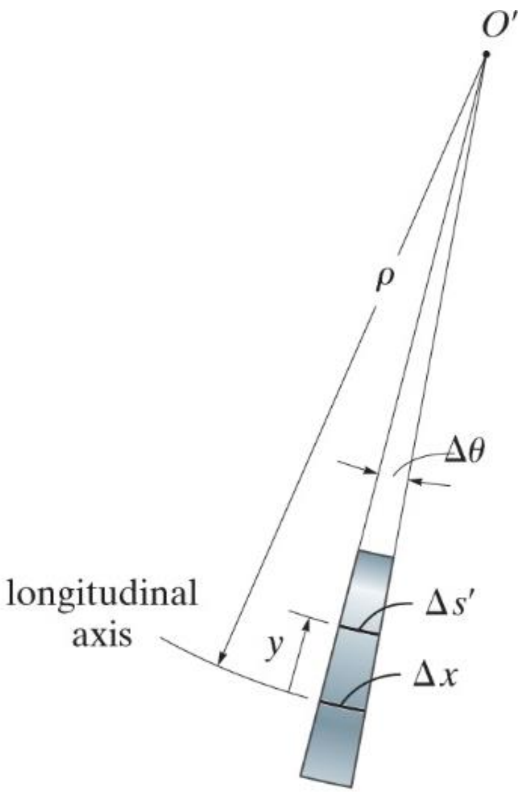
Moment-Curvature Relationship

- It's found that

$$\varepsilon = -\frac{y}{\rho}$$

The Curvature ($1/\rho$)

$$\frac{1}{\rho} = -\frac{\varepsilon}{y}$$



Deformed element

Moment-Curvature Relationship

If material is homogeneous and shows linear-elastic behavior, Hooke's law applies. Since flexure formula also applies, we combing the equations to get

$$\frac{1}{\rho} = \frac{M}{EI}$$

ρ : radius of curvature at a specific point on elastic curve

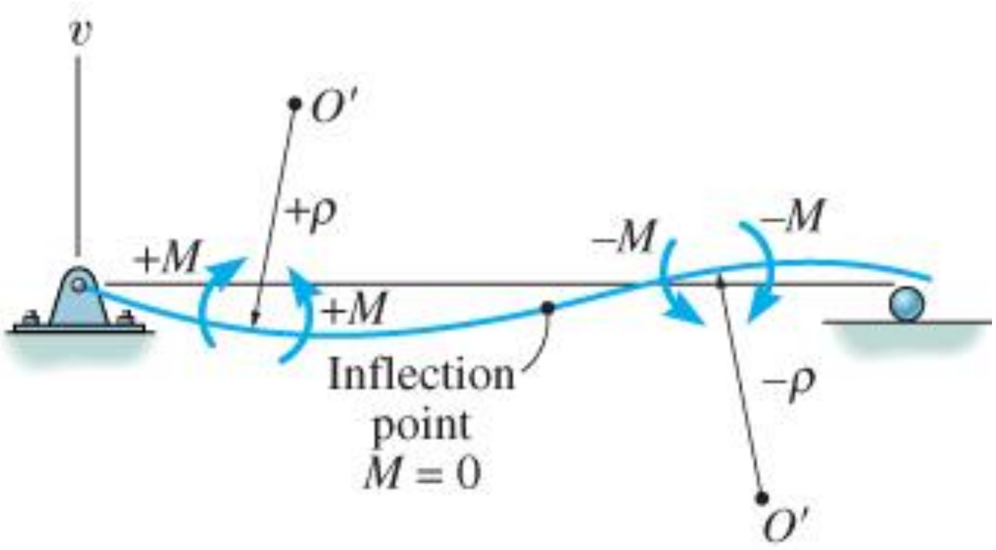
M: internal moment in beam at point where is to be determined.

E : material's modulus of elasticity.

I : beam's moment of inertia computed about neutral axis.

Moment-Curvature Relationship

- EI is the flexural rigidity and is always positive.
- Sign for ρ depends on the direction of the moment.
  - ✓ when M is positive, ρ extends above the beam.
  - ✓ When M is negative, ρ extends below the beam.



Stress-Curvature Relationship

Using flexure formula, curvature is also

$$\left.\begin{array}{l} \dfrac{1}{\rho} = \dfrac{M}{EI} \\ \sigma = -\dfrac{M\,.y}{I} \end{array}\right\} \dfrac{1}{\rho} = -\dfrac{\sigma}{Ey}$$

Moment and Stress-Curvature Relationships are valid for either small or large radii of curvature

Slope and Displacement

Slope and displacement by integration

Slope and displacement by integration

The equation of the elastic curve for a beam can be expressed mathematically as:

$$\upsilon = f\left(x\right)$$

Let's represent the curvature in terms of ν and x.

$$\frac{1}{\rho} = \frac{d^2\upsilon/dx^2}{\left[1+\left(d\upsilon/dx\right)^2\right]^{3/2}}$$

$$\frac{1}{\rho} = \frac{M}{EI} \Rightarrow \frac{d^2\upsilon/dx^2}{\left[1+\left(d\upsilon/dx\right)^2\right]^{3/2}} = \frac{M}{EI}$$

Slope and displacement by integration

Most engineering codes specify limitations on deflections for tolerance or aesthetic purposes.

Slope of elastic curve determined from *dv/dx* is very small and its square will be negligible compared with unity.

Therefore, by approximation

$$\frac{1}{\rho} = \frac{M}{EI} \Rightarrow \frac{d^2\upsilon}{dx^2} = \frac{M}{EI} \Rightarrow M = EI\frac{d^2\upsilon}{dx^2}$$

Differentiate each side with respect to x and substitute $V = dM/dx$, we get

$$V(x) = \frac{d}{dx}\left(EI\frac{d^2\upsilon}{dx^2}\right) = EI\frac{d^3\upsilon}{dx^4}$$

Flexural rigidity is constant along beam

Slope and displacement by integration

Differentiating again with respect to x and substitute *w= dv/dx*, we get

$$w(x) = \frac{d^2}{dx^2}\left(EI\,\frac{d^2\upsilon}{dx^2}\right) = EI\,\frac{d^4\upsilon}{dx^4}$$

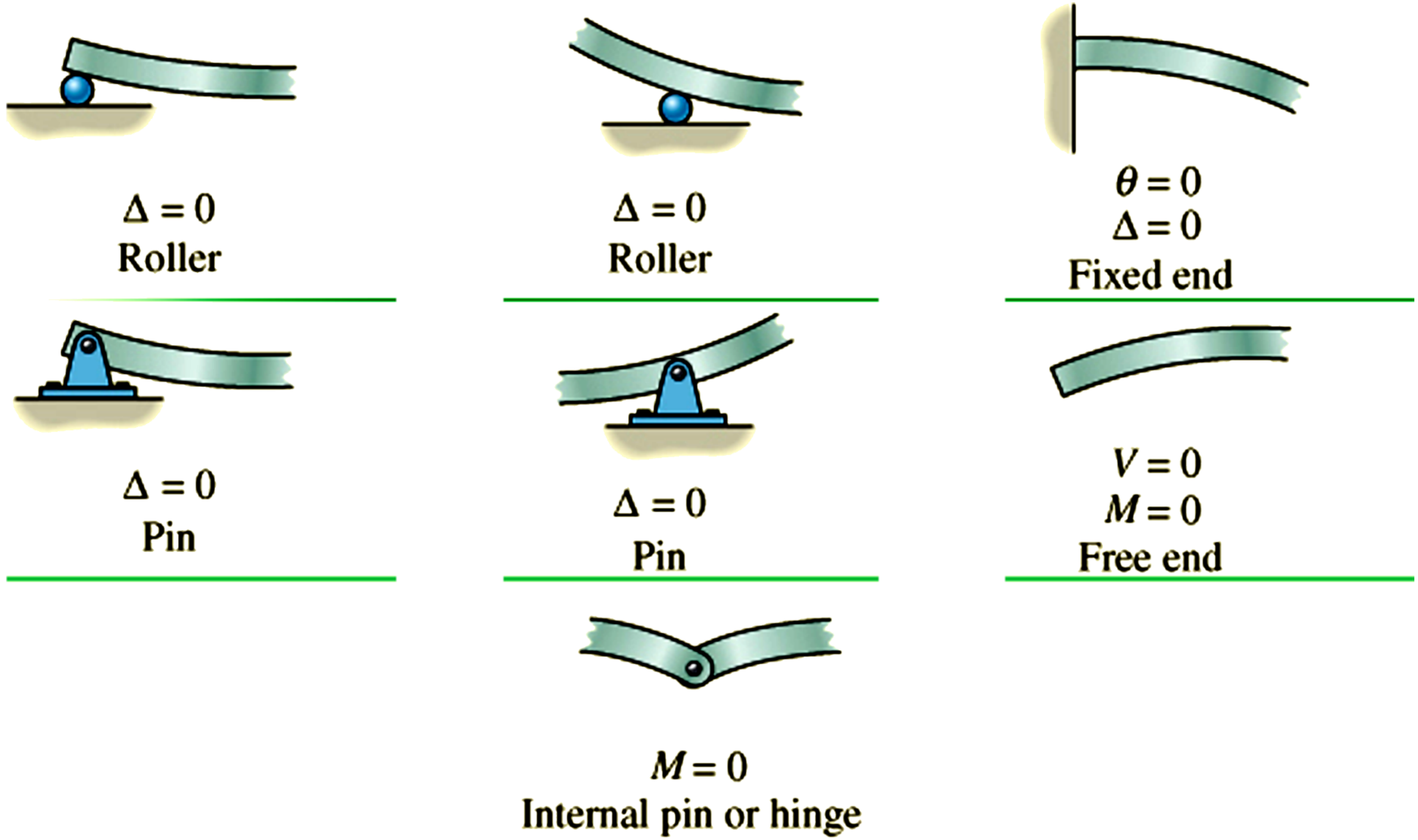
Flexural rigidity is constant along beam

Generally, it is easier to determine the internal moment M as a function of x, integrate twice, and evaluate only two integration constants.

For convenience in writing each moment expression, the origin for each x coordinate can be selected arbitrarily.

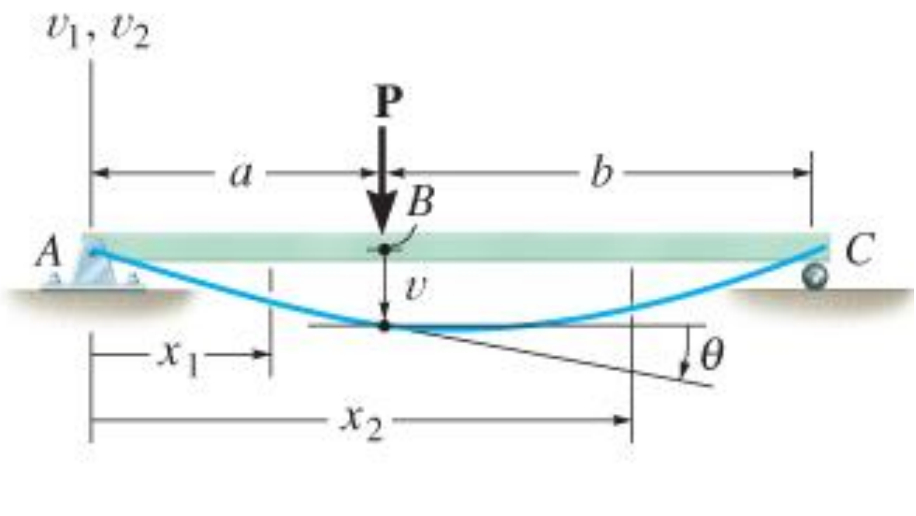
Slope and displacement by integration

Possible boundary conditions are:

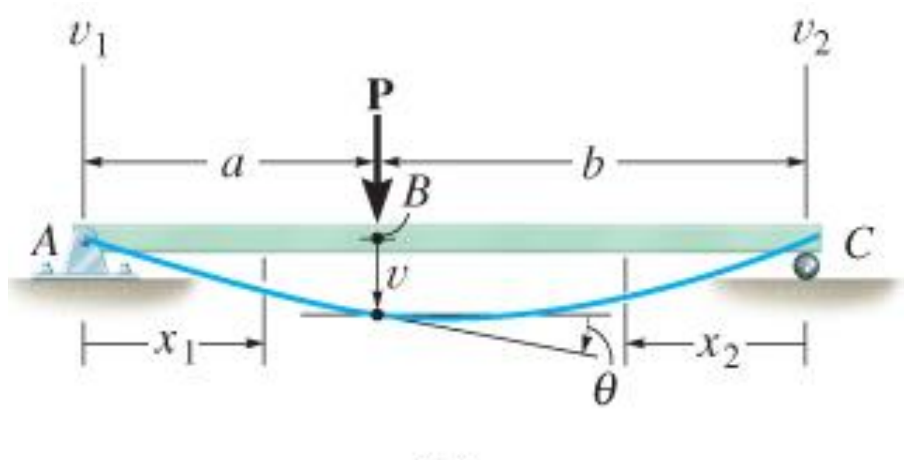


Slope and displacement by integration

If a single *x* coordinate cannot be used to express the eqn for beam's slope or elastic curve, then continuity conditions must be used to evaluate some of the integration constants.



(a)



(b)

Slope and displacement by integration

Procedure for analysis

Elastic curve

Draw an exaggerated view of the beam's elastic curve.

Recall that zero slope and zero displacement occur at all fixed supports, and zero displacement occurs at all pin and roller supports.

Establish the x and v coordinate axes.

The x axis must be parallel to the undeflected beam and can have an origin at any pt along the beam, with +ve direction either to the right or to the left.

Slope and displacement by integration

Procedure for analysis

Elastic curve

If several discontinuous loads are present, establish *x* coordinates that are valid for each region of the beam between the discontinuties.

Choose these coordinates so that they will simplify subsequent algrebraic work.

Slope and displacement by integration

Procedure for analysis

Load or moment function

For each region in which there is an *x* coordinate, express that loading *w* or the internal moment *M* as a function of *x*.

In particular, always assume that *M* acts in the +ve direction when applying the eqn of moment equilibrium to determine $M = f(x)$.

Slope and displacement by integration

Procedure for analysis

Slope and elastic curve

Provided EI is constant, apply either the load eqn $EI\, d^4v/dx^4 = -w(x)$, which requires four integrations to get $v = v(x)$, or the moment eqns $EI\, d^2v/dx^2 = M(x)$, which requires only two integrations. For each integration, we include a constant of integration.

Constants are evaluated using boundary conditions for the supports and the continuity conditions that apply to slope and displacement at pts where two functions meet.

Slope and displacement by integration

Procedure for analysis

Slope and elastic curve

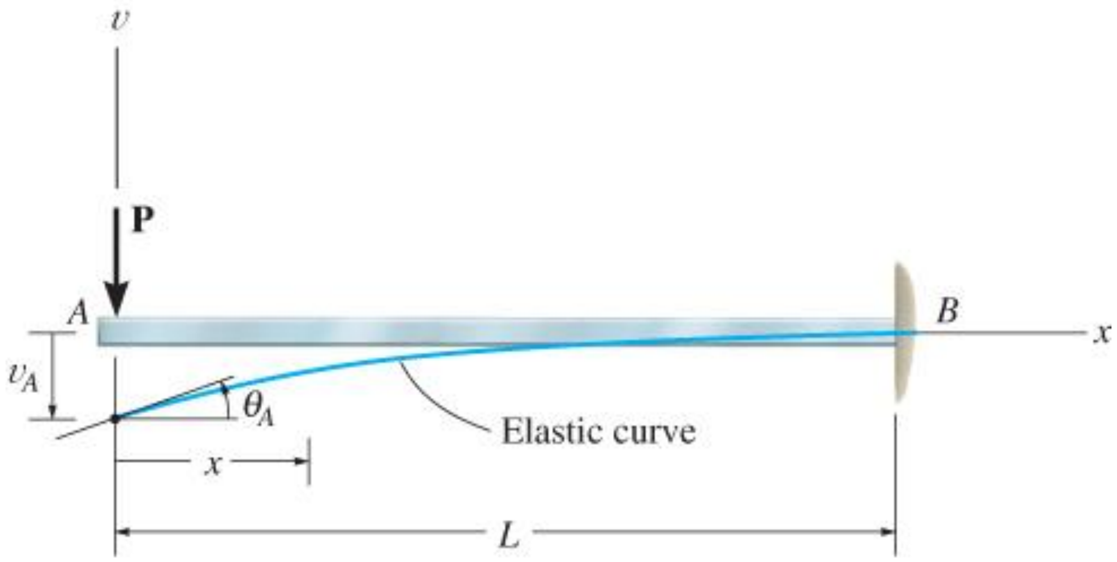
Once constants are evaluated and substituted back into slope and deflection eqns, slope and displacement at specific pts on elastic curve can be determined.

The numerical values obtained is checked graphically by comparing them with sketch of the elastic curve.

Realize that +ve values for slope are counterclockwise if the *x* axis extends +ve to the right, and clockwise if the *x* axis extends +ve to the left. For both cases, +ve displacement is upwards.

EXAMPLE 12.1

Cantilevered beam shown is subjected to a vertical load **P** at its end. Determine the eqn of the elastic



(a)

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Elastic curve
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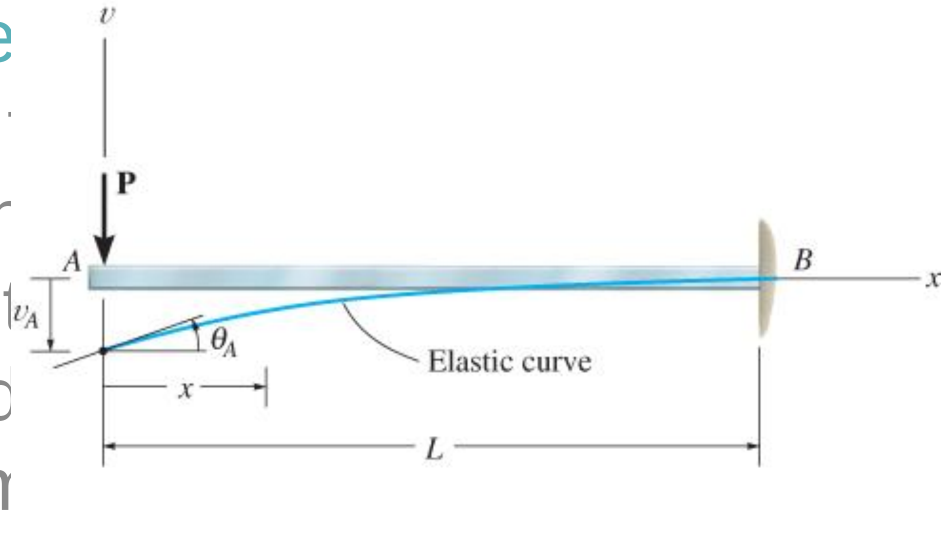
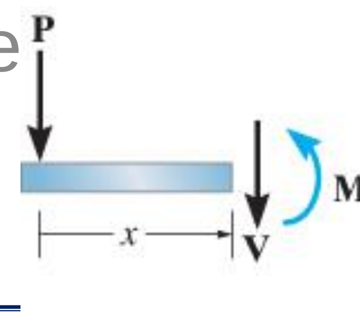


(a)

Moment function: From free-body diagram, with **M** acting in the +ve direction, we

$$M = -Px$$



(b)

Slope and elastic curve:

Applying Eqn 12-10 and integrating twice yields

$$EI\frac{d^2\upsilon}{dx^2}=-Px \qquad (1)$$

$$EI\frac{d\upsilon}{dx}=-\frac{Px^2}{2}+C_1 \qquad (2)$$

$$EI\upsilon=-\frac{Px^3}{6}+C_1x+C_2 \qquad (3)$$

Slope and elastic curve:

Using boundary conditions $dv/dx = 0$ at $x = L$, and $v =$ 0 at $x = L$, Eqn (2) and (3) becomes

$$0 = -\frac{PL^2}{2} + C_1$$

$$0 = -\frac{PL^3}{6} + C_1 L + C_2$$

Slope and elastic curve:

Thus, $C_1 = PL^2/2$ and $C_2 = PL^3/3$. Substituting these results into Eqns (2) and (3) with $\theta = dv/dx$, we get

$$0 = -\frac{P}{2EI}\left(L^2 - x^2\right)$$

$$\upsilon = \frac{P}{6EI}\left(-x^3 + 3L^2 x - 2L^3\right)$$

Maximum slope and displacement occur at A $(x = 0)$,

$$\theta_A = \frac{PL^2}{2EI} \qquad \upsilon_A = -\frac{PL^3}{3EI}$$

Slope and elastic curve:

Positive result for θ_A indicates counterclockwise rotation and negative result for *A* indicates that v_A is downward.

Consider beam to have a length of 5 m, support load *P* = 30 kN and made of A-36 steel having E_{st} = 200 GPa.

Slope and elastic curve:

Using methods in chapter 11.3, assuming allowable normal stress is equal to yield stress σ_{allow} = 250 MPa, then a W310×39 would be adequate
(I = 84.8(10^6) mm^4).

From Eqns (4) and (5),

$$\theta_A = \frac{PL^2}{2EI} \qquad \upsilon_A = -\frac{PL^3}{3EI}$$

Slope and elastic curve:

From Eqns (4) and (5),

$$\theta_A = \frac{30\ \text{kN}\left(10^3\ \text{N/kN}\right)\times\left[5\ \text{m}\left(10^3\ \text{mm/m}\right)^2\right]^2}{2\left[200\left(10^3\right)\text{N/mm}^2\right]\left(84.8\left(10^6\right)\text{mm}^4\right)} = 0.0221\ \text{rad}$$

$$\upsilon_A = -\frac{30\ \text{kN}\left(10^3\ \text{N/kN}\right)\times\left[5\ \text{m}\left(10^3\ \text{mm/m}\right)^2\right]^3}{3\left[200\left(10^3\right)\text{N/mm}^2\right]\left(84.8\left(10^6\right)\text{mm}^4\right)} = -73.7\ \text{mm}$$

Slope and elastic curve:

Since $2\theta_A = (dv/dx)^2 = 0.000488 << 1$, this justifies the use of Eqn 12-10 than the more exact 12-4.

Also, since it is for a cantilevered beam, we've obtained larger values for θ and v than would be obtained otherwise.

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