Lecture 7

Torsion

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Lecture Objectives

- ✓ Discuss effects of applying torsional loading to a long straight member
- ✓ Determine stress distribution within the member under torsional load
- ✓ Determine angle of twist when material behaves in a linear-elastic and inelastic manner

- ✓ Discuss statically indeterminate analysis of shafts and tubes

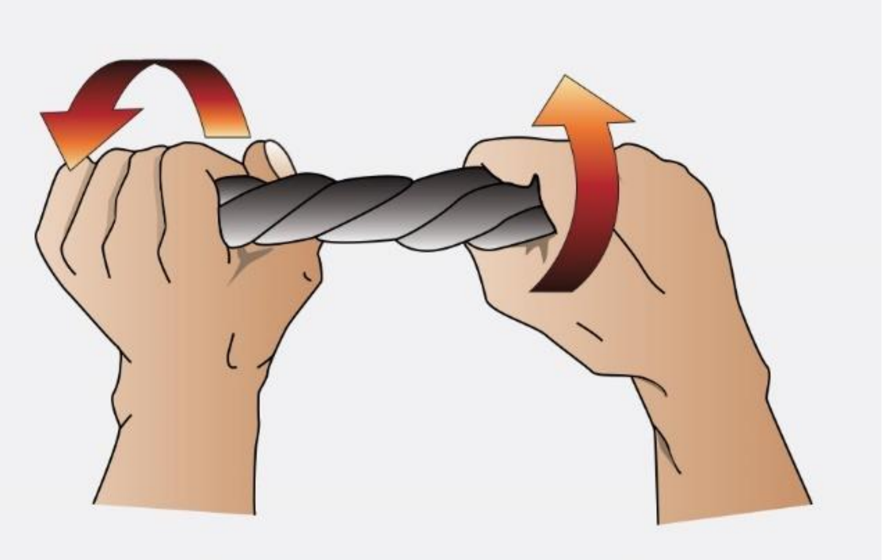
Lecture Outline

- ✓ Torsional Deformation of a Circular Shaft
- ✓ The Torsion Formula
- ✓ Power Transmission
- ✓ Angle of Twist
- ✓ Statically Indeterminate Torque-Loaded Members
- ✓ Solid Noncircular Shafts
- ✓ Thin-Walled Tubes Having Closed Cross Sections

Loading Stats

- Loading stats
  - Axial
    - The load applied along its longitudinal axis
    - Tension
    - Compression
    - Buckling
  - Bending
    - The load applied perpendicular to its longitudinal axis
  - Torsion
    - Torque is a moment that tends to twist a member about its longitudinal axis

Torsion Loading



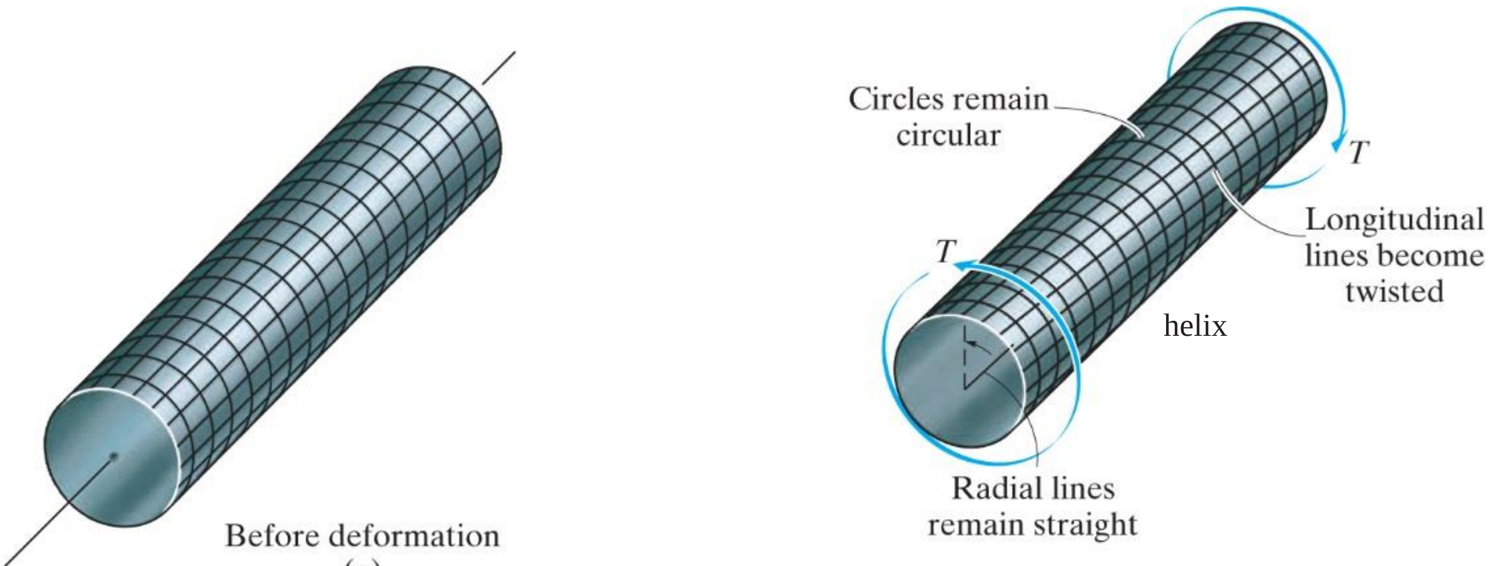
Torsion

Torsion Loading

Torque is a moment that tends to twist a member about its longitudinal axis.

Cross sections from the ends along the shaft will remain flat, they do not warp or bulge in or out.

By observation, if the angle of twist is small, the **length** of the shaft and its **radius** will remain unchanged



Before deformation
(a)

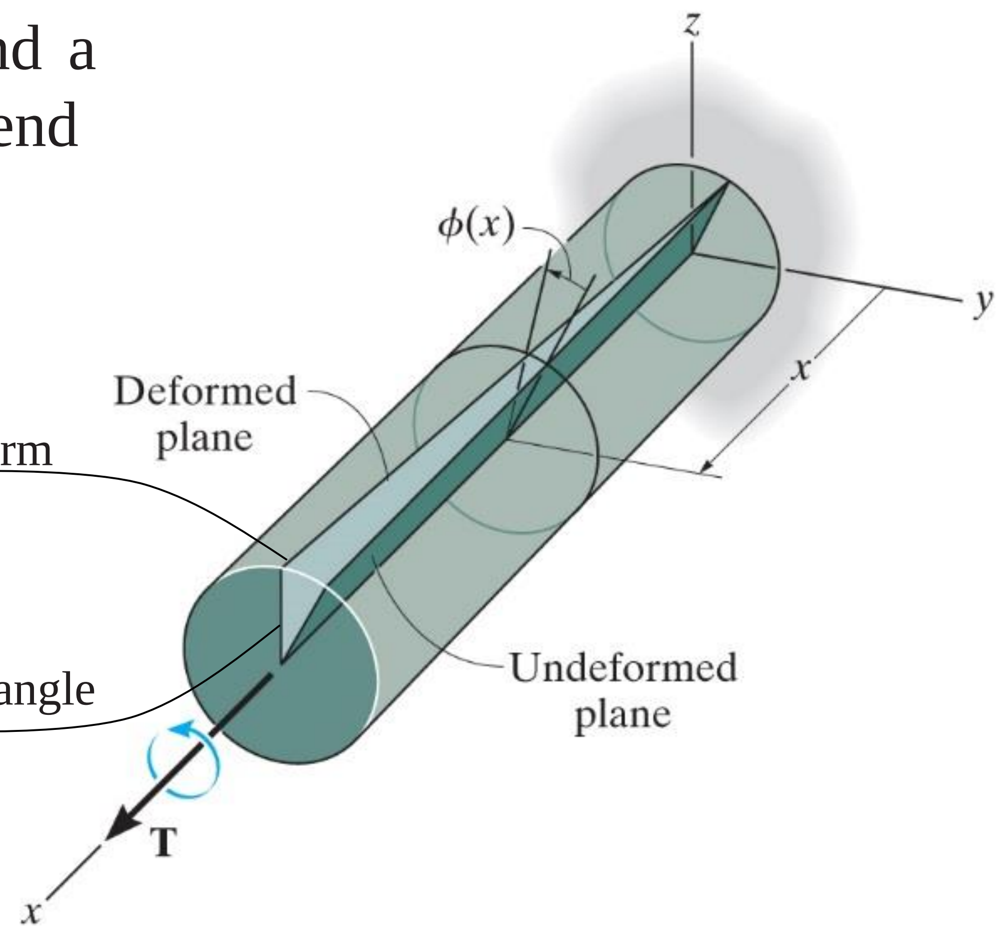
After deformation
(b)

Torsion Loading

Shaft is fixed at one end and a torque is applied to its other end

Plane distort into a skewed form

Radial line rotate through an angle $\phi(x)$ called the angle of twist



The angle of twist $\phi(x)$ increases as x increases.

Torsion Loading

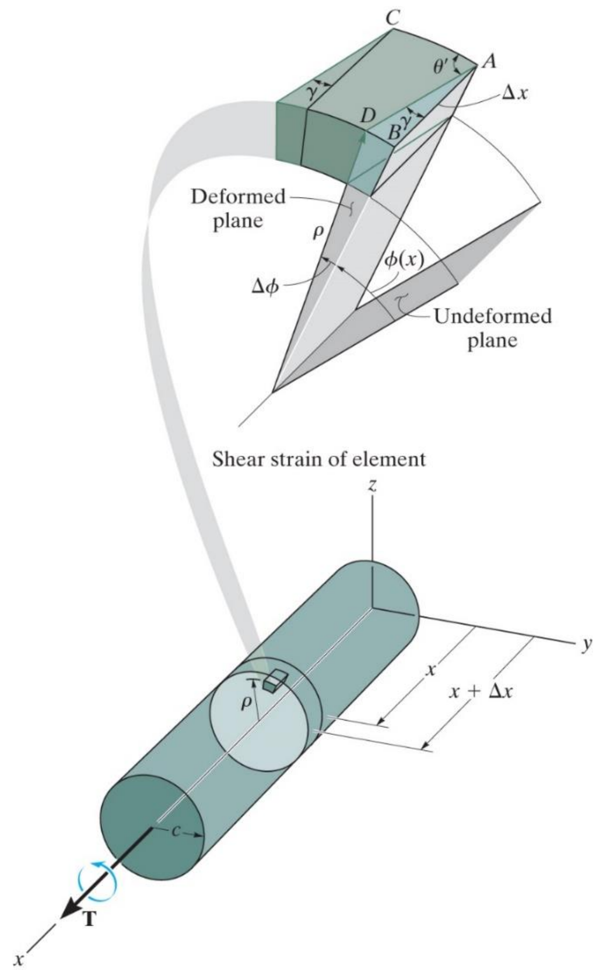
Isolate a small element located at a radial distance (ρ) from the axis

The front and rear faces of the element will undergo a rotation: the back face by $\phi(x)$ and the front face by $\phi(x)+\Delta\phi$.

As a result, the difference in these rotations $\Delta\phi$, causes the element to be subjected to a shear strain.

before deformation the angle between the edges AB and AC is 90°;

after deformation, the angle between the edges AD (deformed edge of AB) and AC is θ';



$$\gamma = \frac{\pi}{2} - \theta'$$

On the other hand

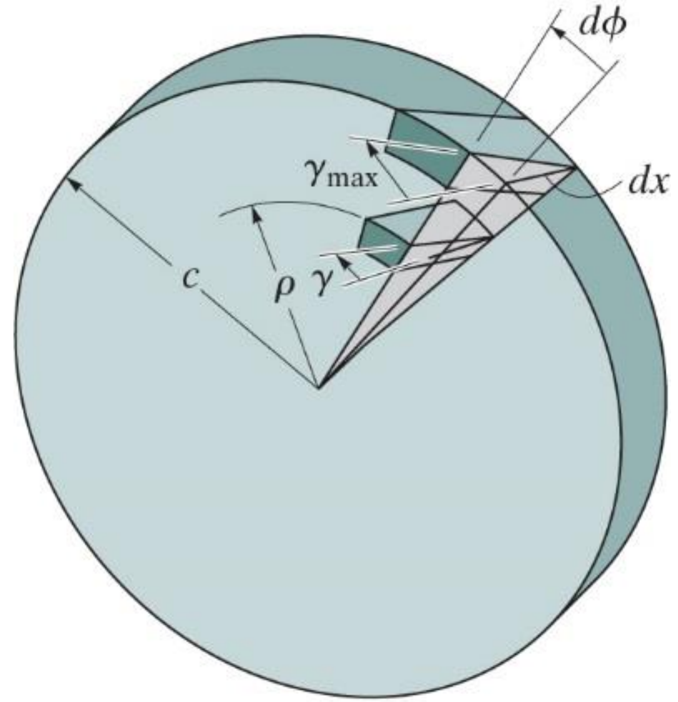
$$BD = \rho.\Delta\phi = \Delta x\,.\gamma \Rightarrow \gamma = \rho\frac{\Delta\phi}{\Delta x} \xrightarrow[\Delta x \to dx]{\Delta\phi \to d\phi} \gamma = \rho\frac{d\phi}{dx}$$

Torsion Loading

$\dfrac{d\phi}{dx}$ is constant over the cross section

$$\frac{d\phi}{dx} = \frac{\gamma}{\rho} = \frac{\gamma_{\max}}{c} \Rightarrow \gamma = \left(\frac{\rho}{c}\right)\gamma_{\max}$$

The results obtained here are also valid for circular tubes.



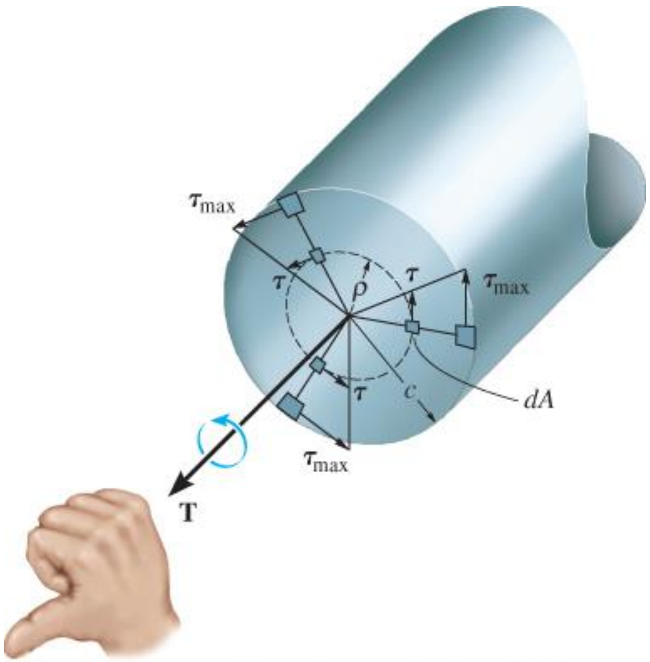
The shear strain at points on the cross section increases linearly with ρ, i.e., $\gamma = (\rho/c)\gamma_{max}$.

The Torsion Formula

If the material is linear-elastic, then Hooke's law applies, and consequently a ***linear variation in shear strain***, leads to a corresponding ***linear variation in shear stress*** along any radial line on the cross section.

$$\tau = \left(\frac{\rho}{c}\right)\tau_{\max}$$

Hence, τ will vary from zero at the shaft's longitudinal axis to a maximum value, at its outer surface.



Shear stress varies linearly along each radial line of the cross section.

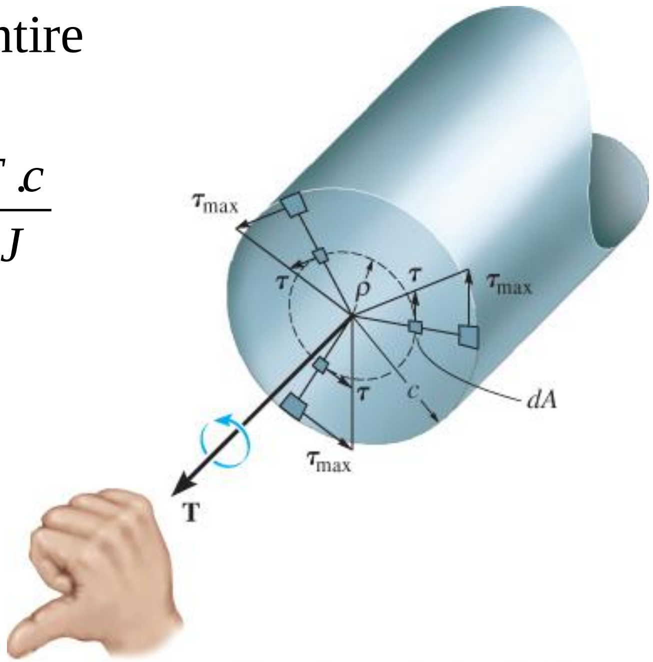
The Torsion Formula

Specifically, each element of area dA , located at ρ is subjected to a force of dF=τ.dA .The torque produced by this force is T= ρ.dF= ρ.τ.dA ,therefore, for the entire cross section

$$T = \int_A \rho.\tau.dA = \int_A \rho.\left(\frac{\rho}{c}\right).\tau_{\max}.dA = \frac{\tau_{\max}}{c}\int_A \rho^2.dA \Rightarrow \tau_{\max} = \frac{T.c}{J}$$

$$j = \int_A \rho^2.dA$$

J: Polar moment of inertia of the shaft's cross-sectional area about the shaft's longitudinal axis



Shear stress varies linearly along each radial line of the cross section.

The Torsion Formula

$$\tau_{\max} = \frac{T.c}{J} \qquad \tau = \frac{T.\rho}{J}$$

used only if :

(1) the shaft is circular;

(2) the material is homogeneous;

(3) behaves in a linear elastic manner.

$\tau_{\max}$	=	the maximum shear stress in the shaft, which occurs at the outer surface
T	=	the resultant *internal torque* acting at the cross section. Its value is determined from the method of sections and the equation of moment equilibrium applied about the shaft's longitudinal axis
J	=	the polar moment of inertia of the cross-sectional area
c	=	the outer radius of the shaft

The Torsion Formula

Polar moment of inertia of the shaft's cross-sectional area about the shaft's longitudinal axis

$$J=\int_A \rho^2.dA=\int_A \rho^2.2.\pi.\rho.d\rho=2.\pi\int_A \rho^3.d\rho$$

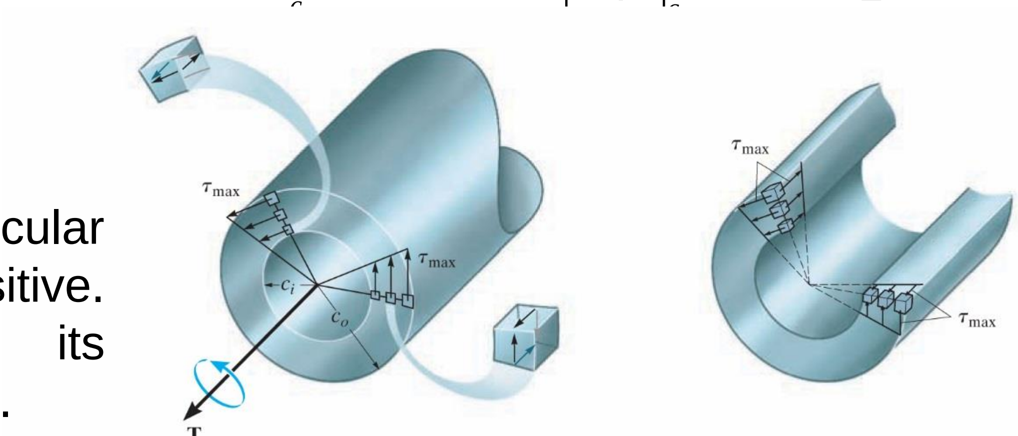
Solid Shaft

$$J=2.\pi\int_0^c \rho^3.d\rho=2.\pi\left[\frac{\rho^4}{4}\right]_0^c=\pi.\frac{c^4}{2}$$

Tubular Shaft.

$$J=2.\pi\int^{c_i} \rho^3.d\rho=2.\pi\left[\frac{\rho^4}{4}\right]^{c_i}=\pi.\frac{c_i{}^4-c_0{}^4}{2}$$

J is a geometric property of the circular area and is always positive. Common units used for its measurement are mm^4 and m^4.



Shear stress varies linearly along each radial line of the cross section.
(a) (b)

Power Transmission

Shafts and tubes having circular cross sections are often used to transmit power developed by a machine. When used for this purpose, they are subjected to a torque that depends on the power generated by the machine and the angular speed of the shaft

$$P = T.\omega = 2.\pi.f.T$$

$$P : [W]$$

$$T : [N.m]$$

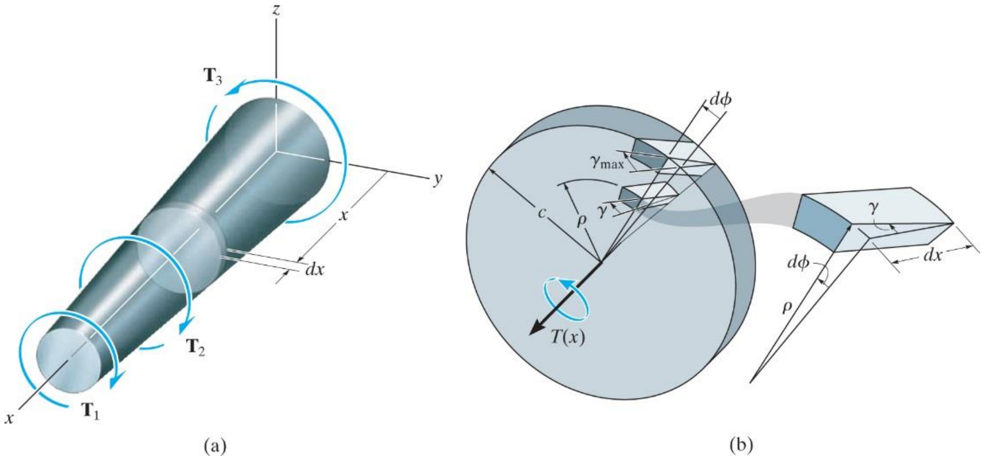
$$f : [s^{-1}], [Hz]$$

$$\omega : [rad]$$

Angle of Twist

Occasionally the design of a shaft depends on restricting the amount of rotation or twist that may occur when the shaft is subjected to a torque.
Furthermore, being able to compute the angle of twist for a shaft is important when analyzing the reactions on statically indeterminate shafts.

$$\gamma = \rho \frac{d\phi}{dx} \Rightarrow d\phi = \frac{\gamma}{\rho} dx$$



$$\tau(\mathrm{x}) = \frac{T(\mathrm{x}).\rho}{J(\mathrm{x})} = G.\gamma(\mathrm{x}) \Rightarrow \gamma(\mathrm{x}) = \frac{T(\mathrm{x}).\rho}{J(\mathrm{x}).G}$$

$$d\phi = \frac{\frac{T(\mathrm{x}).\rho}{J(\mathrm{x}).G}}{\rho} dx = \frac{T(\mathrm{x})}{J(\mathrm{x}).G} dx$$

Angle of Twist

Integrating over the entire length L of the shaft, we obtain the angle of twist for the entire shaft, namely,

$$\phi = \int_0^L \frac{T(\mathrm{x})}{J(\mathrm{x}).G} dx$$

ϕ	= angle of twist, in radians
$T(x)$	= internal torque at arbitrary position x, found from method of sections and equation of moment equilibrium applied about shaft's axis
$J(x)$	= polar moment of inertia as a function of x
G	= shear modulus of elasticity for material

Angle of Twist

Constant Torque and Cross-Sectional Area

In many cases:

- Shaft's cross-sectional area A is constant.
- Material is homogeneous, so G is constant.
- If the external Torque applied at each end is constant
- If the internal Torque along the length of the shaft is constant.

$$\phi = \int_0^L \frac{T\,(\mathrm{x})}{J\,(\mathrm{x}).G}\,dx$$

$$\phi = \frac{T}{J.G}\int_0^L dx$$

$$\phi = \frac{T}{J.G}\left[x\right]_0^L$$

$$\phi = \frac{T.L}{J.G}$$

Angle of Twist

Shaft subjected to several different torques along its length, or the cross-sectional area or shear modulus changes abruptly from one region of the shaft to the next.

The equation $\phi = \frac{T.L}{J.G}$

can be applied to each segment of the shaft where these quantities remain constant.

The angle of twist of one end of the shaft with respect to the other is can be calculated as:

$$\phi = \sum \frac{T.L}{J.G}$$

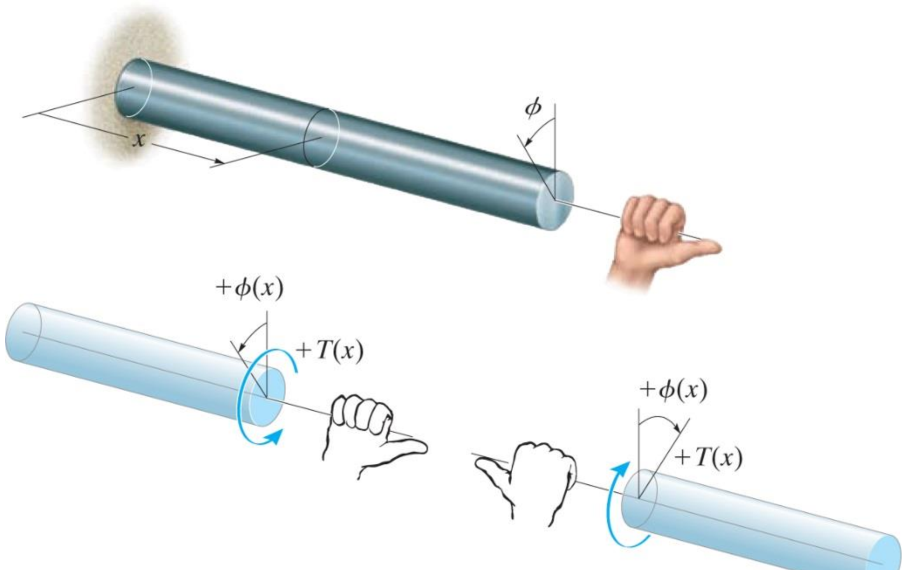
Axial Deformation ⇔ Angle of Twist

$$\delta = \frac{P.L}{E.A}$$

$$\phi = \frac{T.L}{G.J}$$

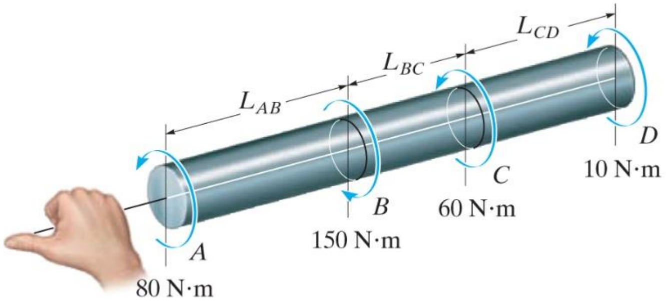
Sign Convention

Using the right-hand rule, the internal torque and angle will be positive, provided the thumb is directed outward from the shaft when the fingers curl to give the tendency for rotation.



Positive sign convention for T and ϕ.

Example (1)

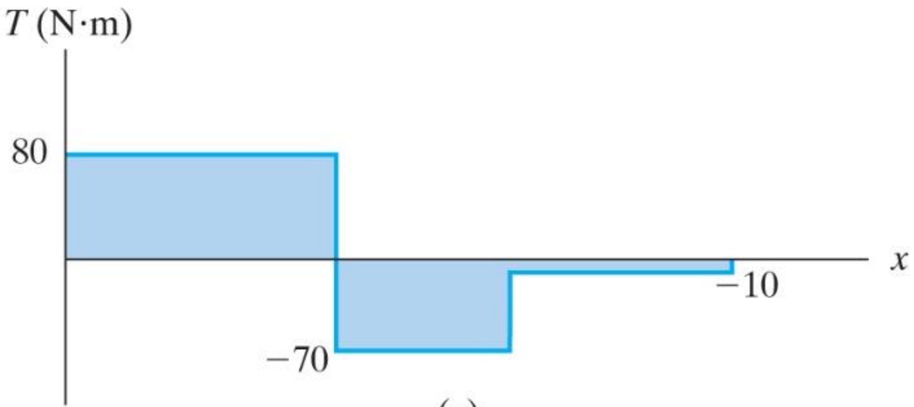


(a)

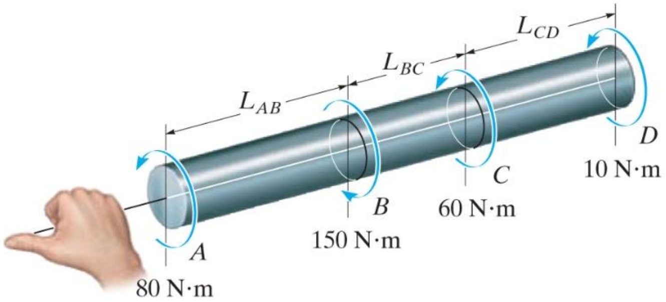
Example (1)

$$\phi = \sum \frac{T.L}{G.j}$$

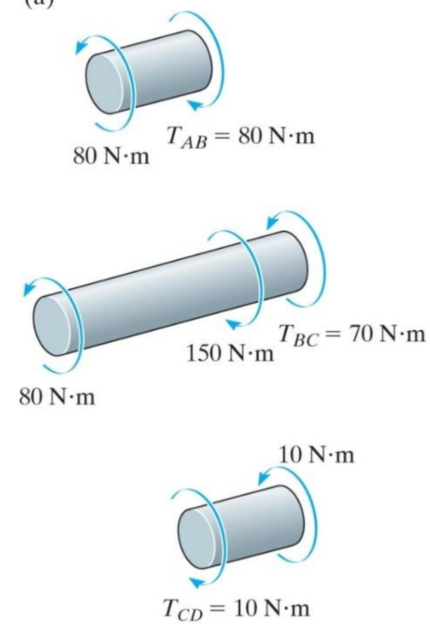
$$\phi_{A/D} = \frac{1}{G.J}\left(80 \times L_{AB} - 70 \times L_{BC} - 10 \times L_{CD}\right)$$



(c)

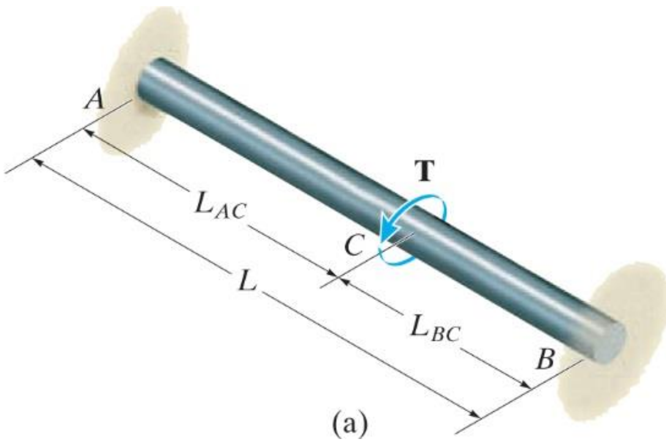


(a)



(b)

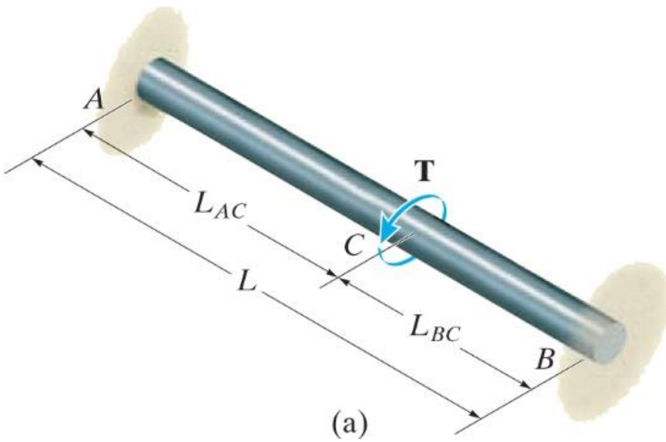
Statically Indeterminate Torque-Loaded Members



Consider a bar which is fixed supported at both of its ends.

How it can be analyzed ?

Statically Indeterminate Torque-Loaded Members

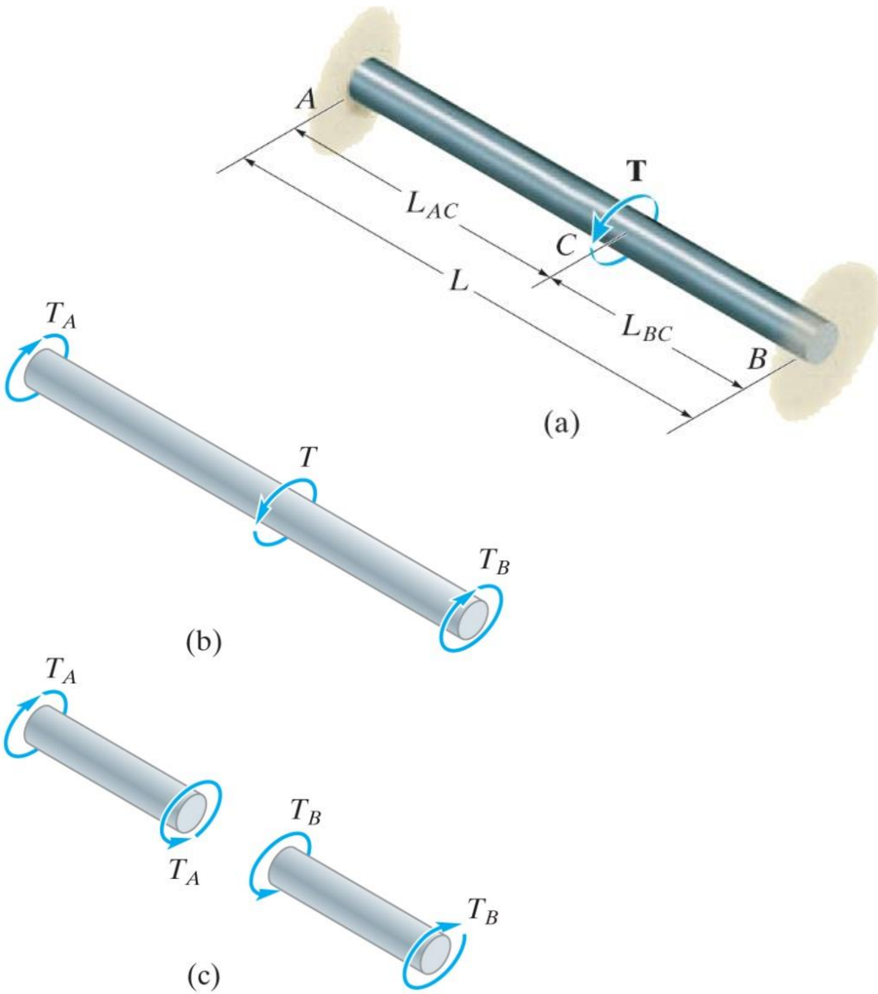


The equilibrium equation(s) are not sufficient to determine the two reactions on the bar.

This type of problem is called **statically indeterminate**

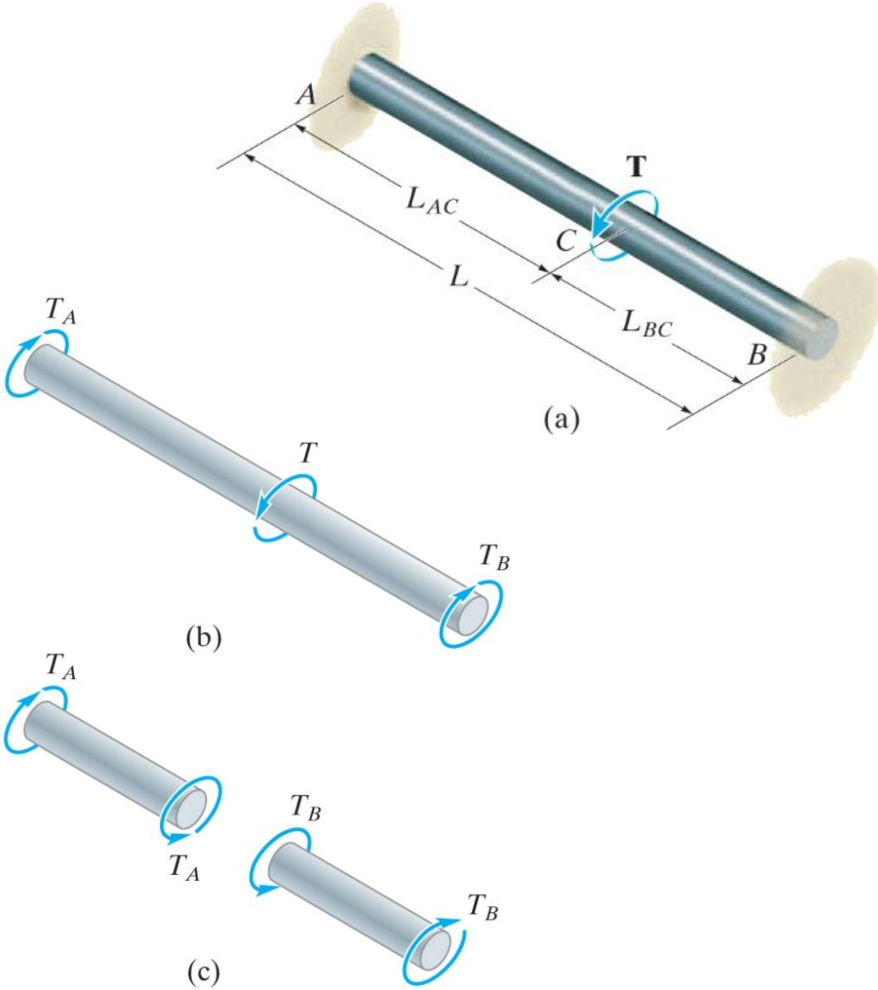
Statically Indeterminate Torque-Loaded Members

A torsionally loaded shaft may be classified as statically indeterminate if the moment equation of equilibrium, applied about the axis of the shaft, is not adequate to determine the unknown torques acting on the shaft.



Statically Indeterminate Torque-Loaded Members

In order to establish an additional equation needed for solution, it is necessary to consider how points on the shaft twist. Specifically, an equation that specifies the conditions for twist is referred to as a compatibility or kinematic condition. In this case, a suitable compatibility condition would require the angle of twist of one end of the shaft with respect to the other end to be equal to zero.



Statically Indeterminate Torque-Loaded Members

$$\sum M = 0; \qquad T - T_A - F_B = 0$$

$$\phi_{A/B} = 0$$

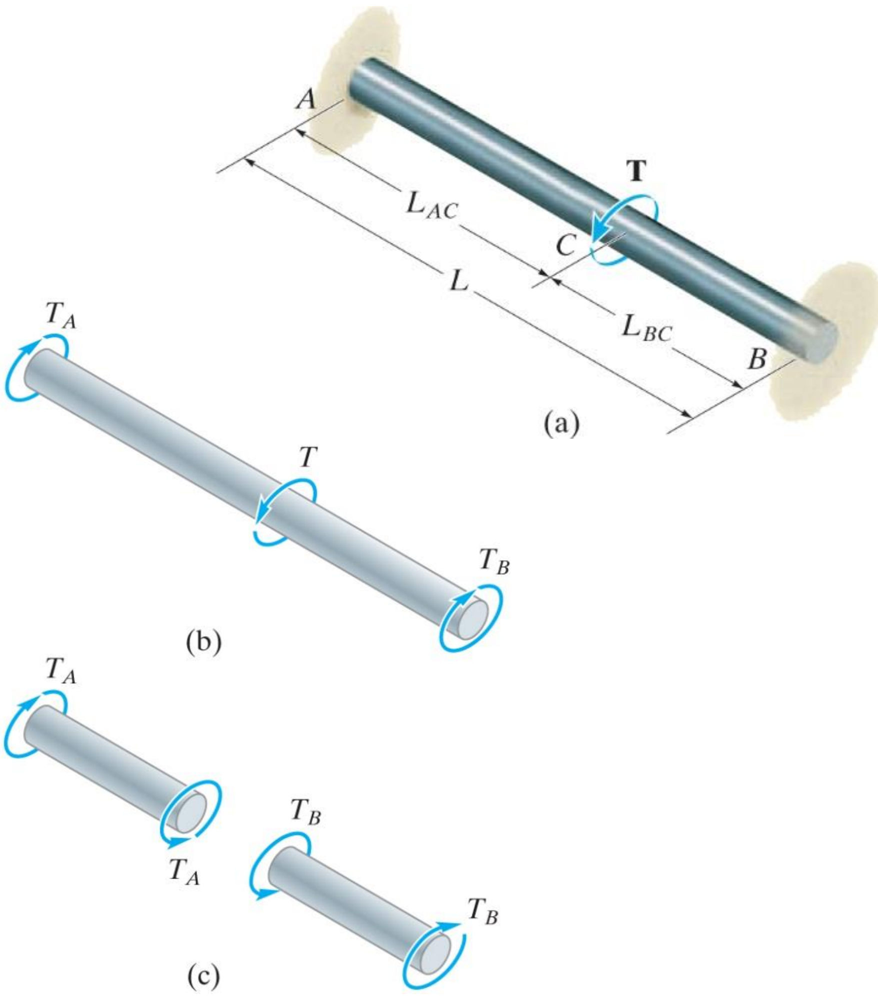
$$\iota_{A/B} = \frac{T_A . L_{AC}}{G.J} - \frac{T_B . L_{CB}}{G.J} = 0$$

Assuming that *J and G* is constant

$$T_A = \frac{T_B . L_{CB}}{L_{AC}}$$

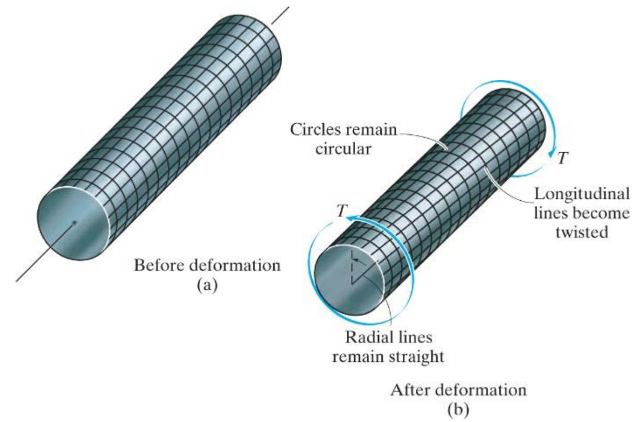
Applying T_A in the equation of equilibrium

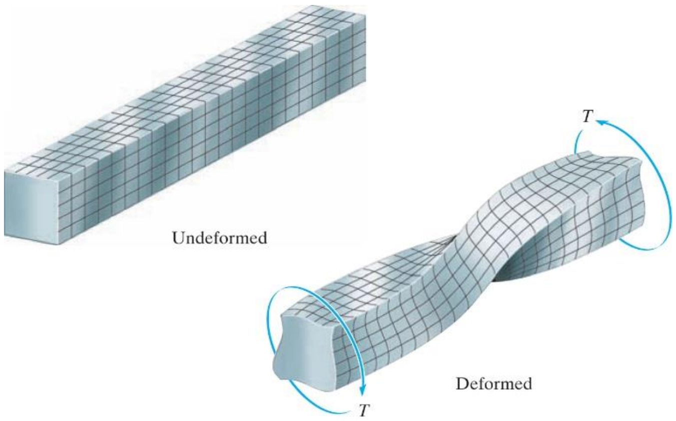
$$T - \frac{T_B . L_{CB}}{L_{AC}} - T_B = 0 \Rightarrow T_B = T\,\frac{L_{AC}}{L} \Rightarrow T_A = T\,\frac{L_{CB}}{L}$$



Solid Noncircular Shafts

For a shaft having a circular cross section, due to the uniformity of the shear strain at all points on the same radius, the cross sections do not deform, Shafts that have a noncircular cross section, however, are not axisymmetric, and so their cross sections will bulge or warp when the shaft is twisted. Evidence of this can be seen from the way grid lines deform on a shaft having a square cross section when the shaft is twisted.



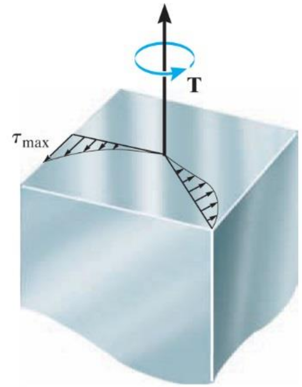


Solid Noncircular Shafts

Using a mathematical analysis based on the theory of elasticity, however, it is possible to determine the shear-stress distribution within a shaft of square cross section.

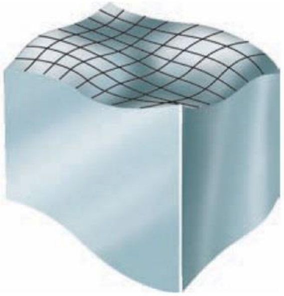
Because these shear-stress distributions vary in a complex manner, the shear strains will warp the cross section.

In particular notice that the corner points of the shaft must be subjected to zero shear stress and therefore zero shear strain.



Shear stress distribution along two radial lines

(a)


Warping of cross-sectional area

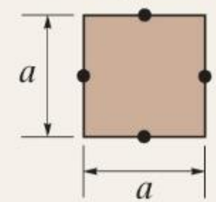
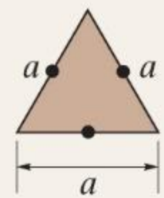
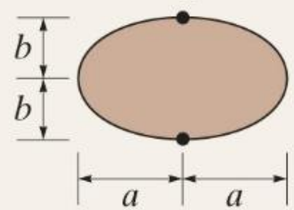
(b)

Solid Noncircular Shafts

$$\tau_{max} = \frac{16T}{\pi . a^3} = \frac{5.09T}{a^3} \qquad \phi = \frac{32T.L}{\pi . a^4 . G} = \frac{10.18T.L}{a^4 . G}$$

Circular cross section

A shaft having a circular cross section is most efficient, since it is subjected to both a smaller maximum shear stress and a smaller angle of twist than a corresponding shaft having a noncircular cross section and subjected to the same torque.

Shape of cross section	τ_{max}	ϕ
Square 	$\frac{4.81\,T}{a^3}$	$\frac{7.10\,TL}{a^4 G}$
Equilateral triangle 	$\frac{20\,T}{a^3}$	$\frac{46\,TL}{a^4 G}$
Ellipse 	$\frac{2\,T}{\pi a b^2}$	$\frac{(a^2+b^2)TL}{\pi a^3 b^3 G}$