Mechanics of Materials

Lecture 6

Axial loading

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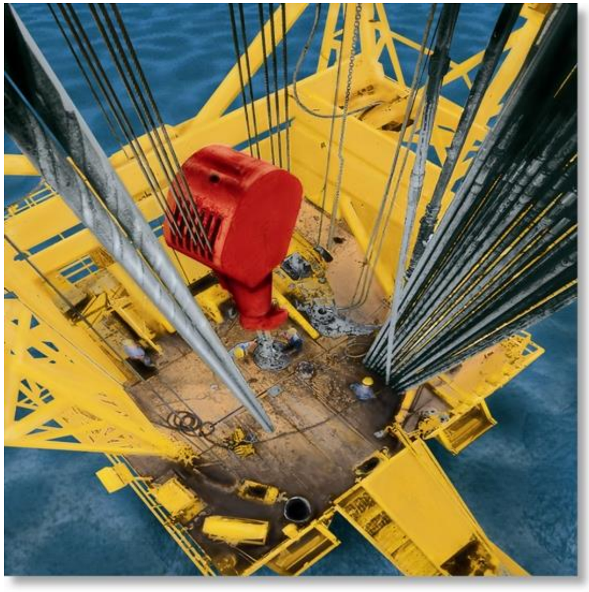
Lecture Objectives

- ✓ Determine deformation of axially loaded members.
- ✓ Develop a method to find support reactions when it cannot be determined from equilibrium equations.



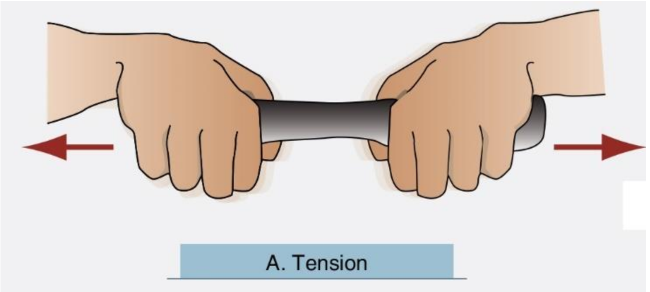
- ✓ Analyze the effects of thermal stress.

Lecture Outline

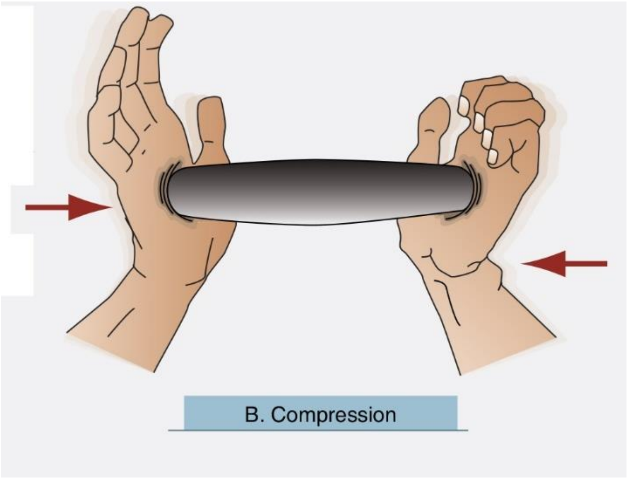
- ✓ Loading Stats
- ✓ Saint-Venant's Principle
- ✓ Elastic Deformation of an Axially Loaded Member
- ✓ Statically Indeterminate Axially Loaded Member
- ✓ Principle of Superposition
- ✓ Statically Indeterminate Axially Loaded Member
- ✓ Force Method of Analysis for Axially Loaded Member
- ✓ Thermal Stress

Loading Stats

Axial



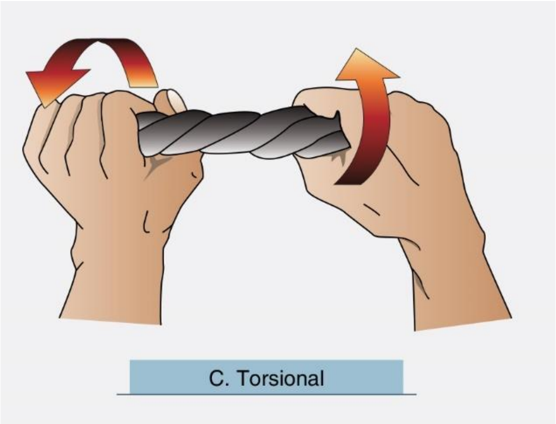
Tension



Compression

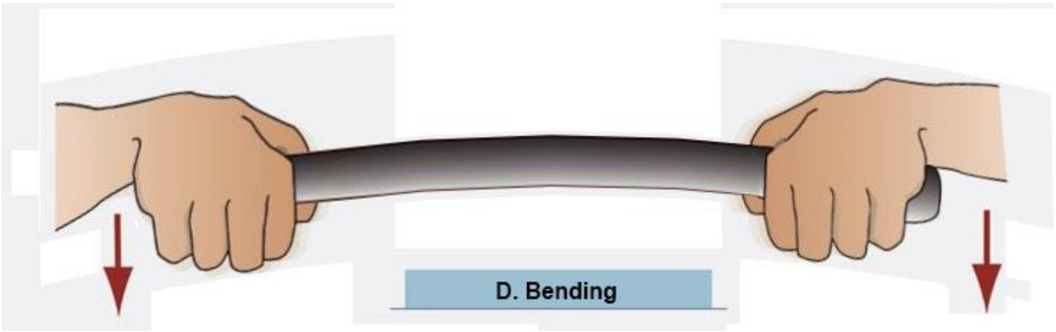
Loading Stats

Torsion



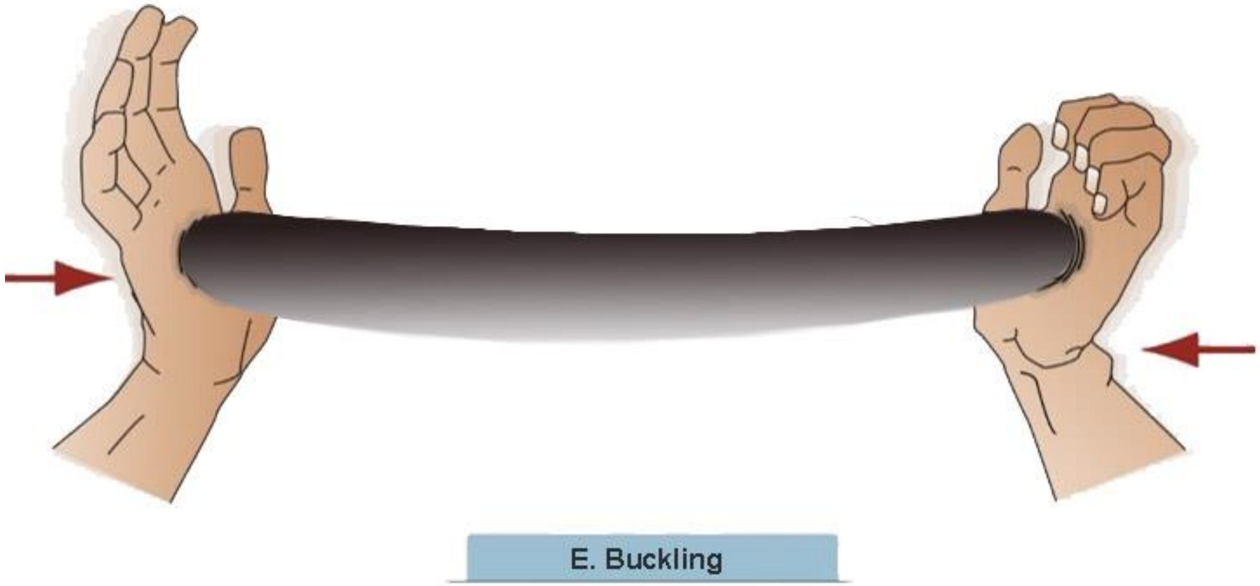
Loading Stats

Bending



Loading Stats

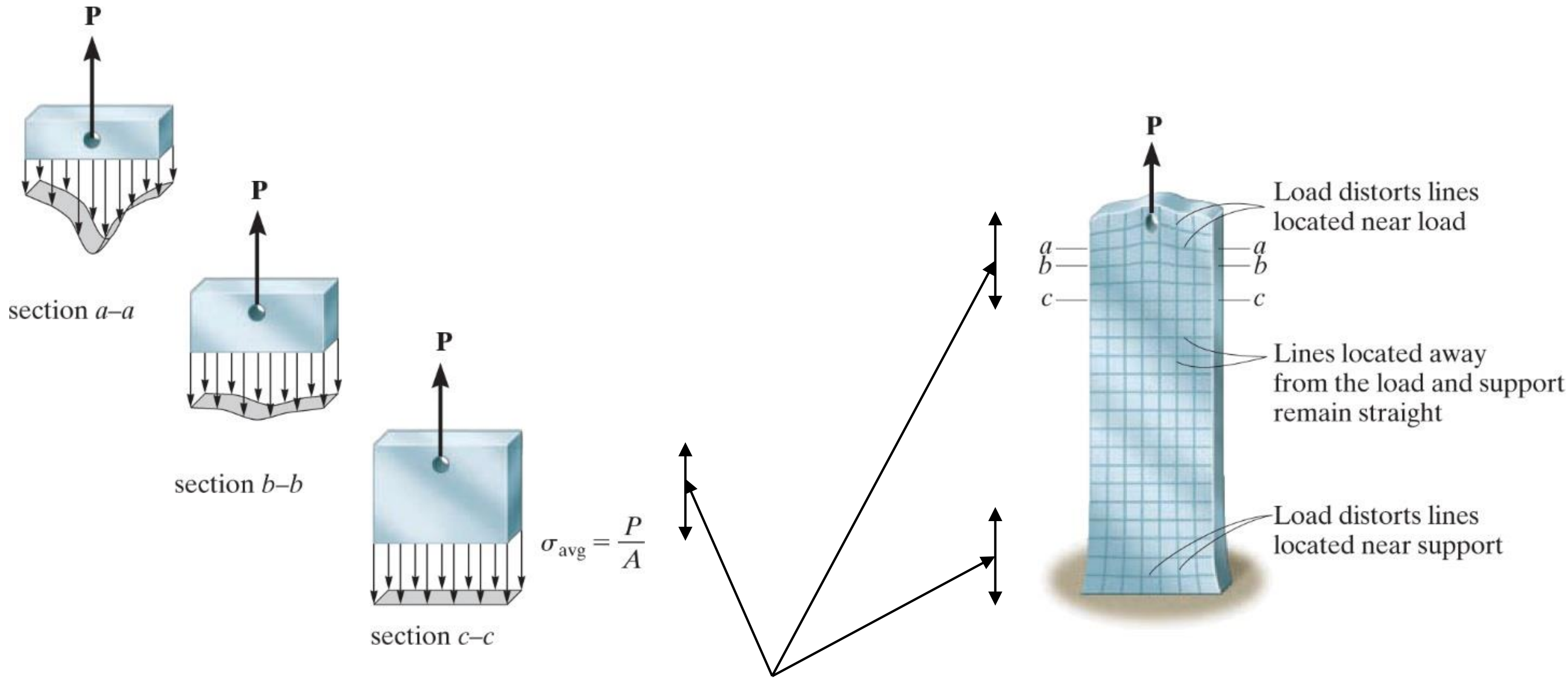
Buckling



Loading Stats

- Loading stats
  - Axial
    - The load applied along its longitudinal axis
    - Tension
    - Compression
    - Buckling
  - Bending
    - The load applied perpendicular to its longitudinal axis
  - Torsion
    - Torque is a moment that tends to twist a member about its longitudinal axis

Saint – Venant's Princible



Mathematical analysis based on the theory of elasticity

Minimum distance from the bar's end where this occurs should at least be equal to the *largest dimension* of the loaded cross section

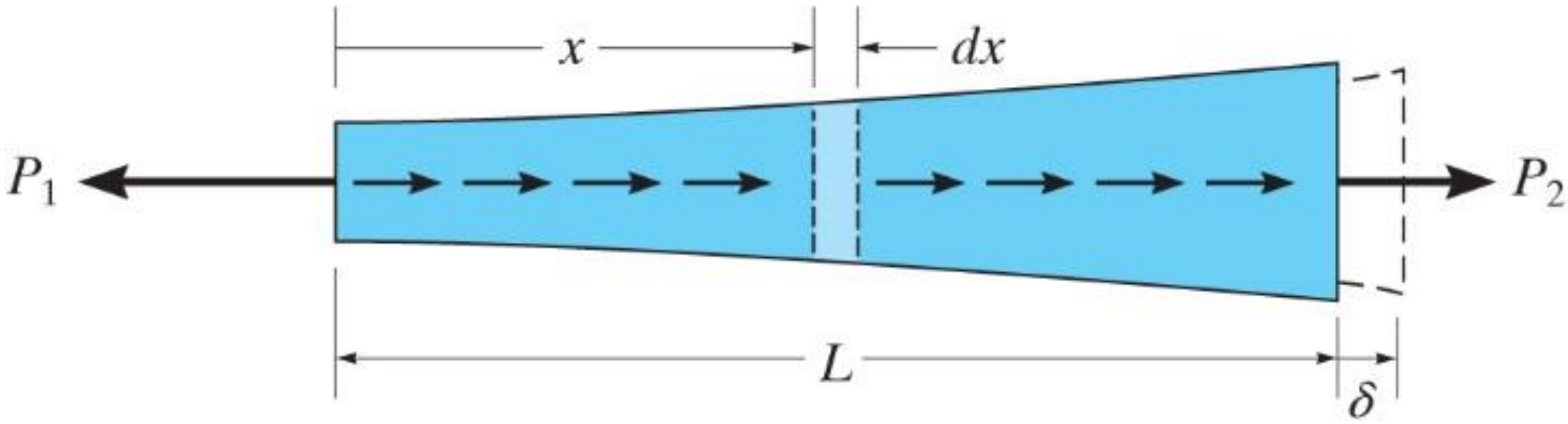
Saint – Venant's Principle

- General rule: min. distance is at least equal to largest dimension of loaded x-section. For the bar, the min. distance is equal to width of bar
- This behavior discovered by Barré de Saint-Venant in 1855, this the name of the principle
- Saint-Venant Principle states that localized effects caused by any load acting on the body, will dissipate/smooth out within regions that are sufficiently removed from location of load
- Thus, no need to study stress distributions at that points near application loads or support reactions

Elastic Deformation of an Axially Loaded Member

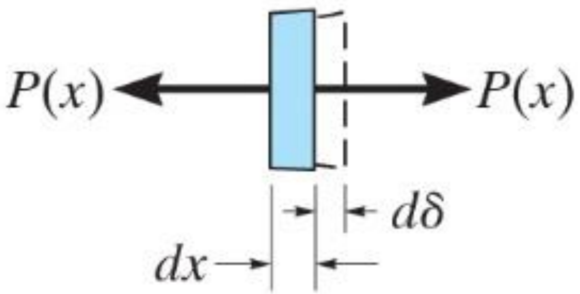
Consider a bar which has a cross-sectional area A(x) that gradually varies along its length L,

Assume the bar is subjected to concentrated loads P_1 and P_2 at its ends and a variable external load distributed along its length.



Elastic Deformation of an Axially Loaded Member

To find the relative displacement δ (delta) of one end of the bar with respect to the other end as caused by this loading, consider a differential element of length dx and cross-sectional area A(x) from the bar at the arbitrary position x.

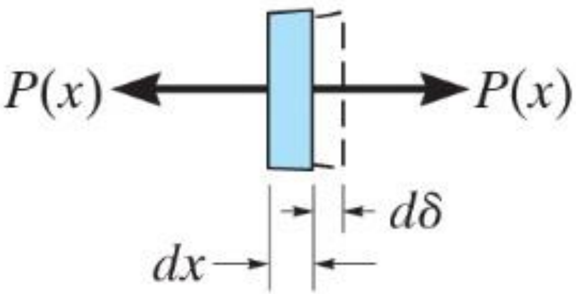


The resultant internal axial force P(x) will be a function of x, and it will deform the element into the shape indicated by the dashed outline, and therefore the displacement of one end of the element with respect to the other end is dδ

Elastic Deformation of an Axially Loaded Member

The stress and strain in the element are:

$$\sigma = \frac{P(x)}{A(x)} \qquad \varepsilon = \frac{d\delta}{dx}$$

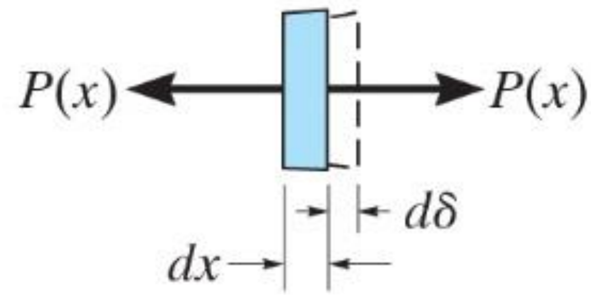


Provided the stress does not exceed the proportional limit, we can apply Hooke's law:

$$\sigma = E\,\varepsilon \Leftrightarrow \frac{P(x)}{A(x)} = E\,\frac{d\delta}{dx} \Rightarrow d\delta = \frac{P(x)}{E.A(x)}dx$$

Elastic Deformation of an Axially Loaded Member

For the entire length L of the bar, The displacement of one point on the bar relative to the other point



$$\delta = \int_0^L \frac{P(x)}{E.A(x)} dx$$

δ = displacement of one point relative to another point
L = distance between the two points
P(x) = internal axial force at the section, located a distance x
A(x) = x-sectional area of the bar, expressed as a function of x
E = modulus of elasticity for material

Elastic Deformation of an Axially Loaded Member

Constant Load and Cross-Sectional Area

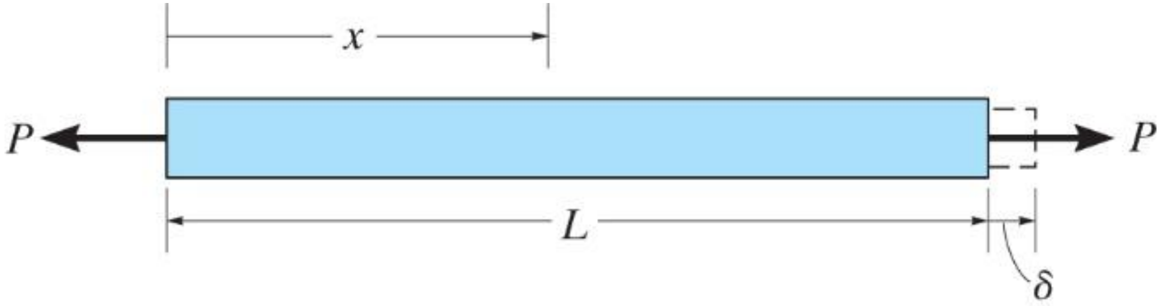
In many cases:

- Cross-sectional area A is constant.
- Material is homogeneous, so E is constant.
- If the external force applied at each end is constant
- If the internal force throughout the length of the bar is constant.

$$\delta = \int_0^L \frac{P}{E.A} dx$$

$$\delta = \left[\frac{P}{E.A} x \right]_0^L$$

$$\delta = \frac{P.L}{E.A}$$



Elastic Deformation of an Axially Loaded Member

Bar is subjected to several different axial forces along its length, or the cross-sectional area or modulus of elasticity changes abruptly from one region of the bar to the next.

The equation

$$\delta = \frac{P.L}{E.A}$$

can be applied to each segment of the bar where these quantities remain constant.

The displacement of one end of the bar with respect to the other is can be calculated as:

$$\delta = \sum \frac{P.L}{E.A}$$

Elastic Deformation of an Axially Loaded Member

Sign Convention

Sign	Forces	Displacement
Positive (+)	Tension	Elongation
Negative (−)	Compression	Contraction

Elastic Deformation of an Axially Loaded Member

Procedure for analysis

Internal force

- ✓ Use method of sections to determine internal axial force P in the member
- ✓ If the force varies along member's strength, section made at the arbitrary location x from one end of member and force represented as a function of x, i.e., P(x)
- ✓ If several constant external forces act on member, internal force in each segment, between two external forces, must then be determined.
- ✓ For any segment, internal tensile force is positive and internal compressive force is negative. Results of loading can be shown graphically by constructing the normal-force diagram.

Elastic Deformation of an Axially Loaded Member

Procedure for analysis

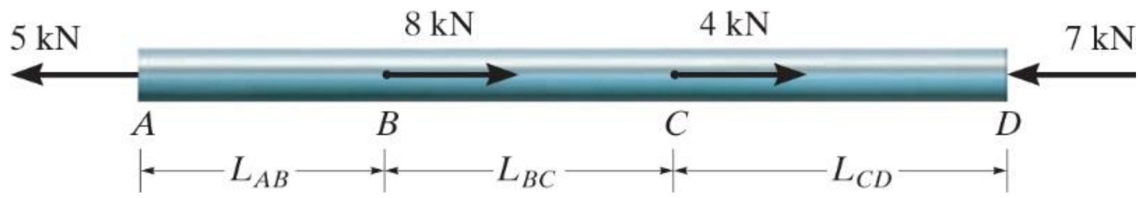
Displacement

- ✓ When member's x-sectional area varies along its axis, the area should be expressed as a function of its position x, i.e., A(x).
- ✓ If x-sectional area, modulus of elasticity, or internal loading suddenly changes, then the equation $\delta = \frac{P.L}{E.A}$ should be applied to each segment for which the quantity are constant
- ✓ When substituting data into equations, account for proper sign for P, tensile loadings positive, compressive negative. Use consistent set of units. If result is positive, elongation occurs, negative means it's a contraction.

Elastic Deformation of an Axially Loaded Member

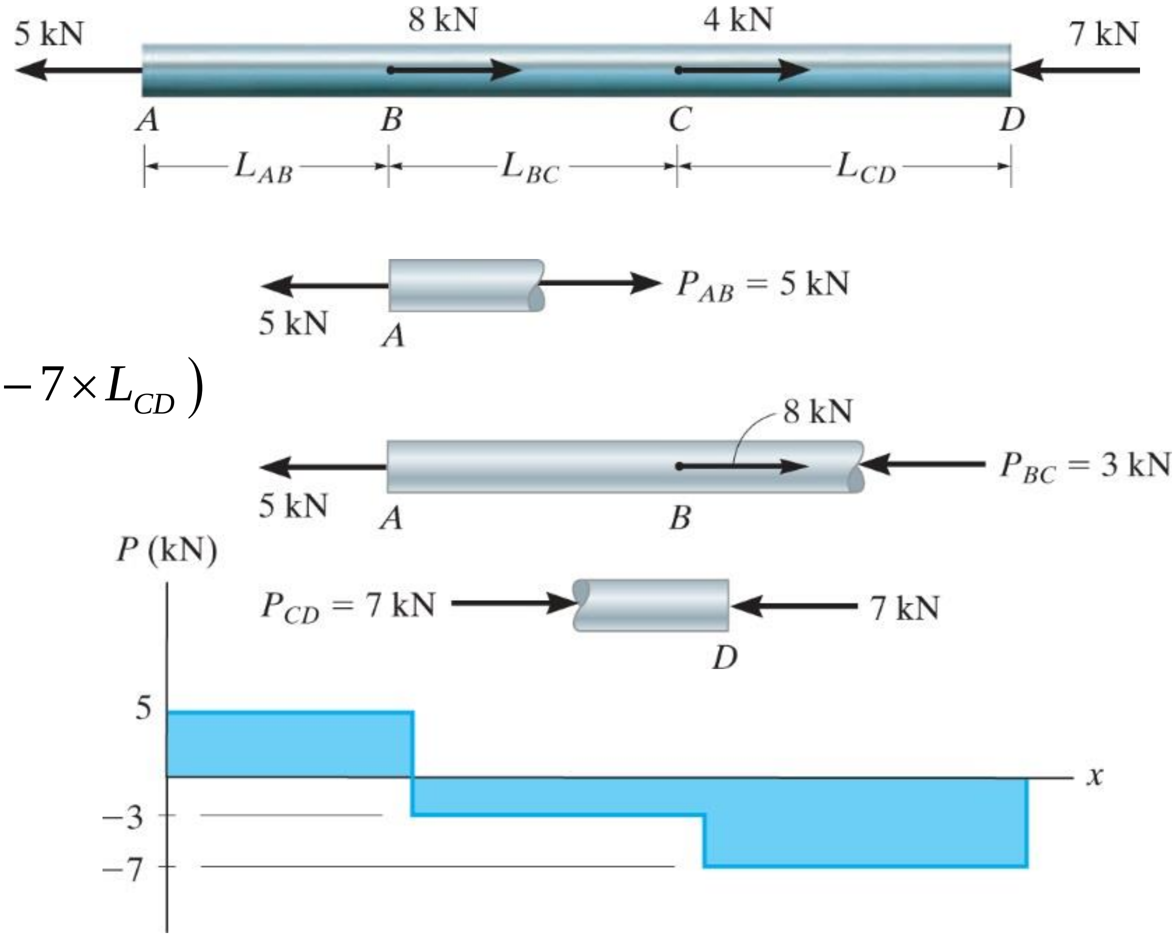
Example:

For the steel bar shown, Determine the displacement of end D relative to *A*.



Elastic Deformation of an Axially Loaded Member

Example:



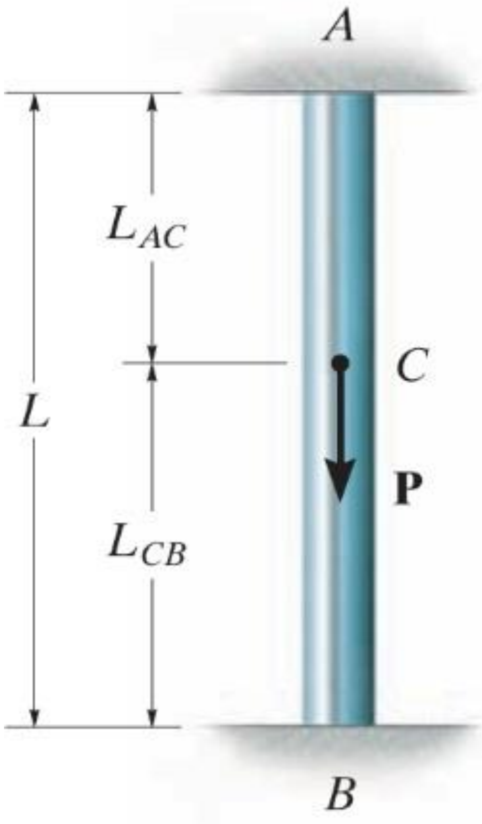
$$\delta = \sum \frac{P.L}{E.A}$$

$$\delta_{A/D} = \frac{1}{E.A}\left(5 \times L_{AB} - 3 \times L_{BC} - 7 \times L_{CD}\right)$$

Statically Indeterminate Axially Loaded Member

Consider a bar which is fixed supported at both of its ends.

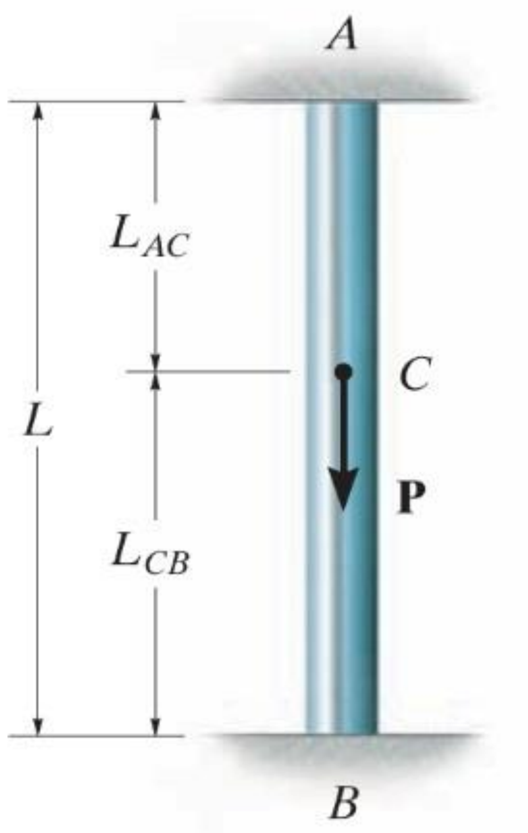
How it can be analyzed ?



Statically Indeterminate Axially Loaded Member

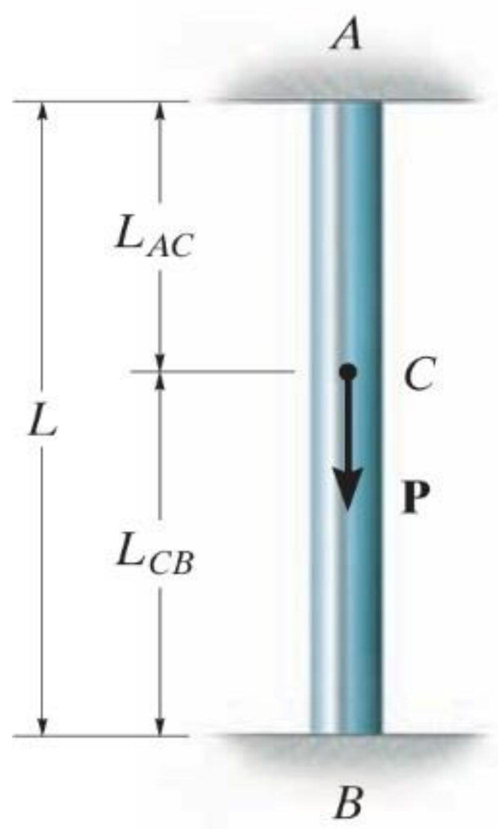
The equilibrium equation(s) are not sufficient to determine the two reactions on the bar.

This type of problem is called **statically indeterminate**



Statically Indeterminate Axially Loaded Member

In order to establish an additional equation needed for solution, it is necessary to consider how points on the bar displace. Specifically, an equation that specifies the conditions for displacement is referred to as a compatibility or kinematic condition. In this case, a suitable compatibility condition would require the displacement of one end of the bar with respect to the other end to be equal to zero.



Statically Indeterminate Axially Loaded Member

$$+\uparrow \sum F = 0; \qquad F_A + F_B - P = 0$$

$$\delta_{A/B} = 0$$

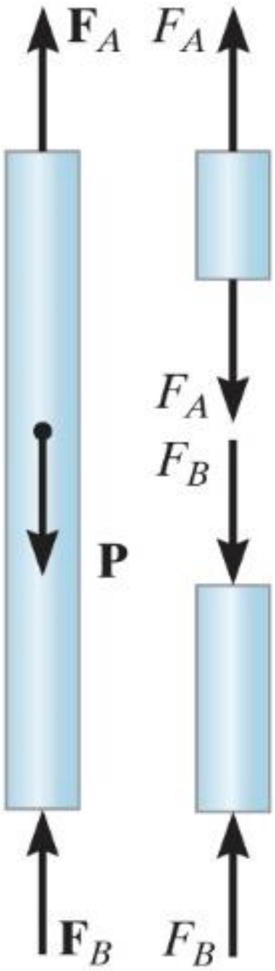
$$\delta_{A/B} = \frac{F_A . L_{AC}}{E . A} - \frac{F_B . L_{CB}}{E . A} = 0$$

Assuming that $A \ and \ E$ is constant

$$F_A = \frac{F_B . L_{CB}}{L_{AC}}$$

Applying F_A in the equation of equilibrium

$$\frac{F_B . L_{CB}}{L_{AC}} + F_B - P = 0$$



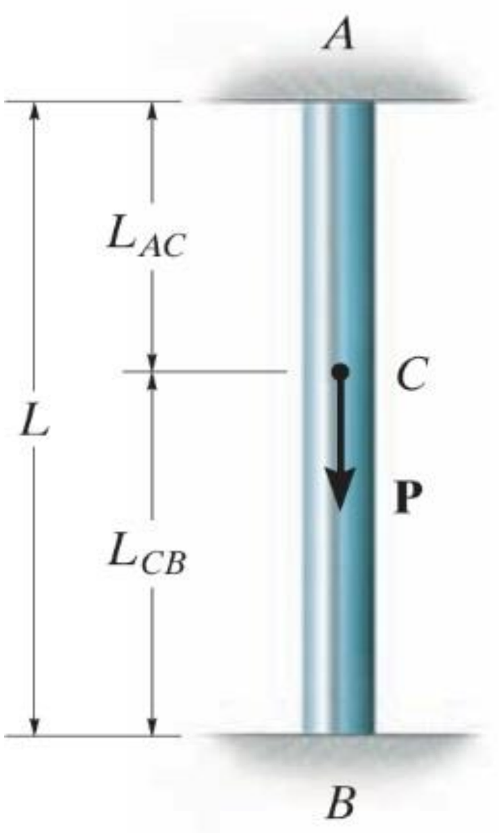
Statically Indeterminate Axially Loaded Member

$$\frac{F_B . L_{CB}}{L_{AC}} + \frac{F_B . L_{AC}}{L_{AC}} - P = 0$$

$$F_B (L_{CB} + L_{AC}) = P . L_{AC}$$

$$F_B = \frac{P . L_{AC}}{(L_{CB} + L_{AC})} = P \frac{L_{AC}}{L}$$

$$\Rightarrow F_A = P \frac{L_{AC}}{L} \frac{L_{CB}}{L_{AC}} = P \frac{L_{CB}}{L}$$



Principle of Superposition

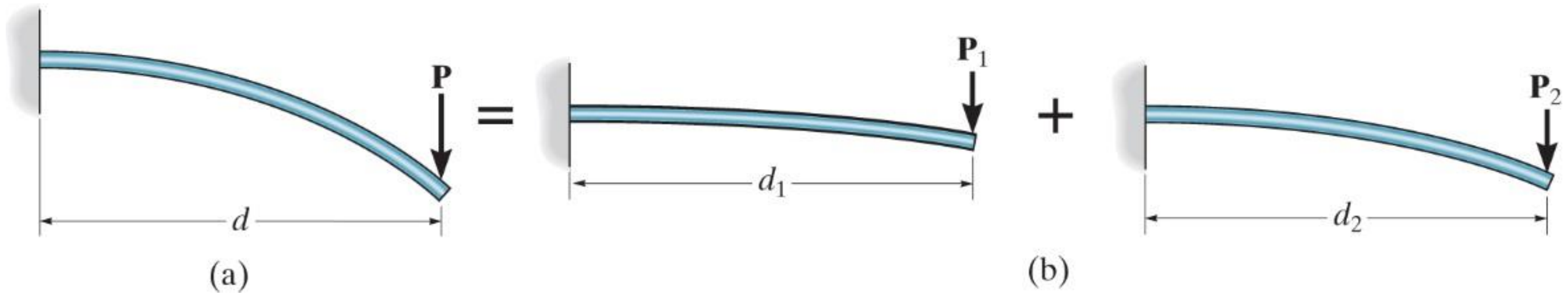
The principle of superposition is often used to determine the stress or displacement at a point in a member when the member is subjected to a complicated loading.

By subdividing the loading into components, the principle of superposition states that the resultant stress or displacement at the point can be determined by algebraically summing the stress or displacement caused by each load component applied separately to the member.

Principle of Superposition

The following two conditions must be satisfied if the principle of superposition is to be applied.

1. The loading must be linearly related to the stress or displacement that is to be determined.
2. The loading must not significantly change the original geometry or configuration of the member.



The Force Method for Axially Loaded Members

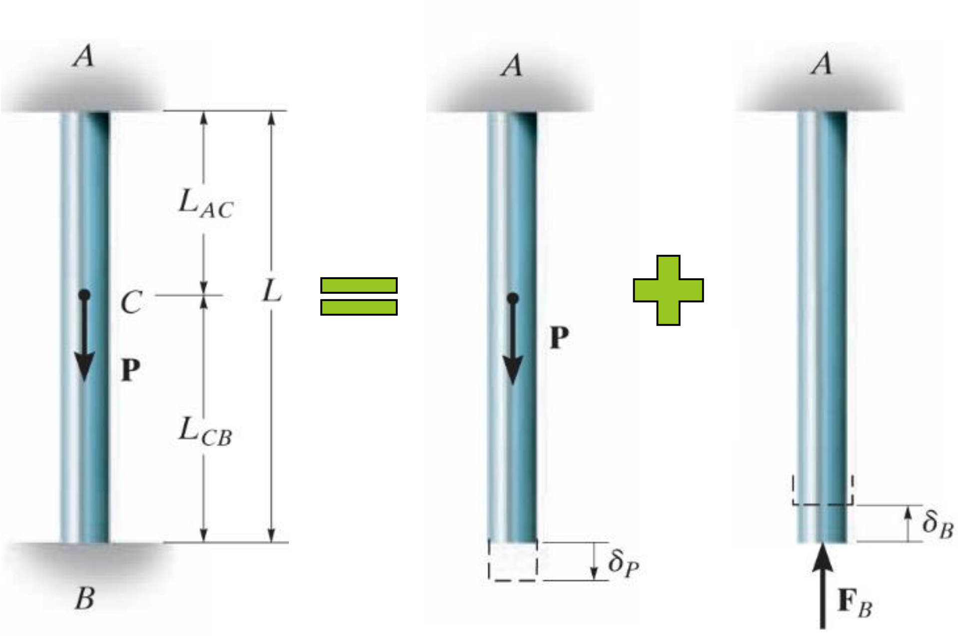
It is possible to solve statically indeterminate problems by writing the compatibility equation using the principle of superposition. This method of solution is often referred to as the flexibility or force method of analysis.

(1) = (2) + (3)

(1) No displacement at *B*

(2) Displacement at *B* when redundant force at *B* is removed

(3) Displacement at *B* when only the redundant force at *B* is applied



The Force Method for Axially Loaded Members

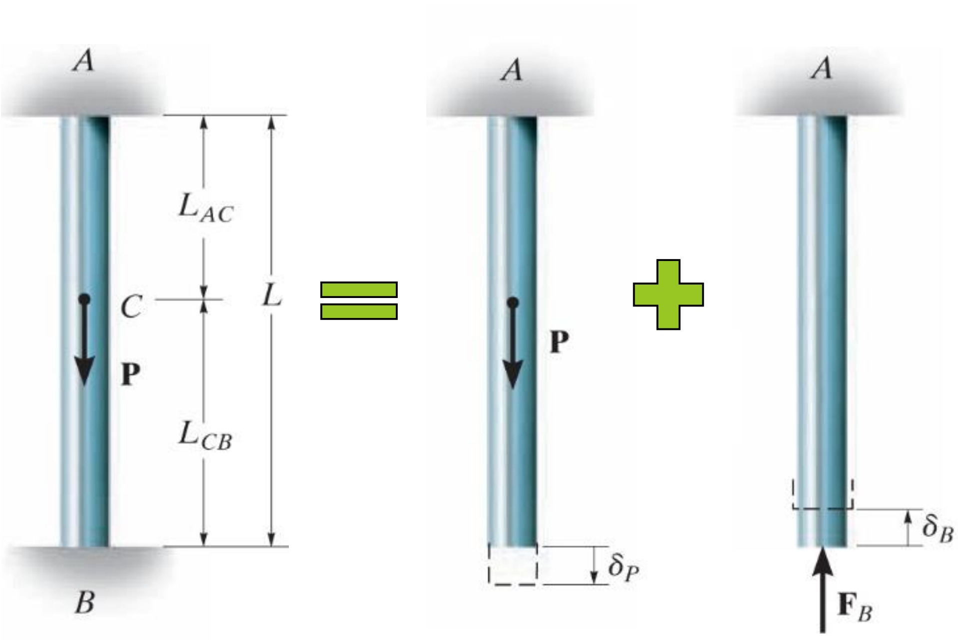
The compatibility equation for displacements at point B, for which we have assumed that displacements are positive downward.

$$\delta_P - \delta_B = 0$$

Applying the load–displacement relationship to each case

$$\frac{P.L_{AC}}{E.A} - \frac{F_B.L}{E.A} = 0$$

$$F_B = \frac{P.L_{AC}}{L}$$



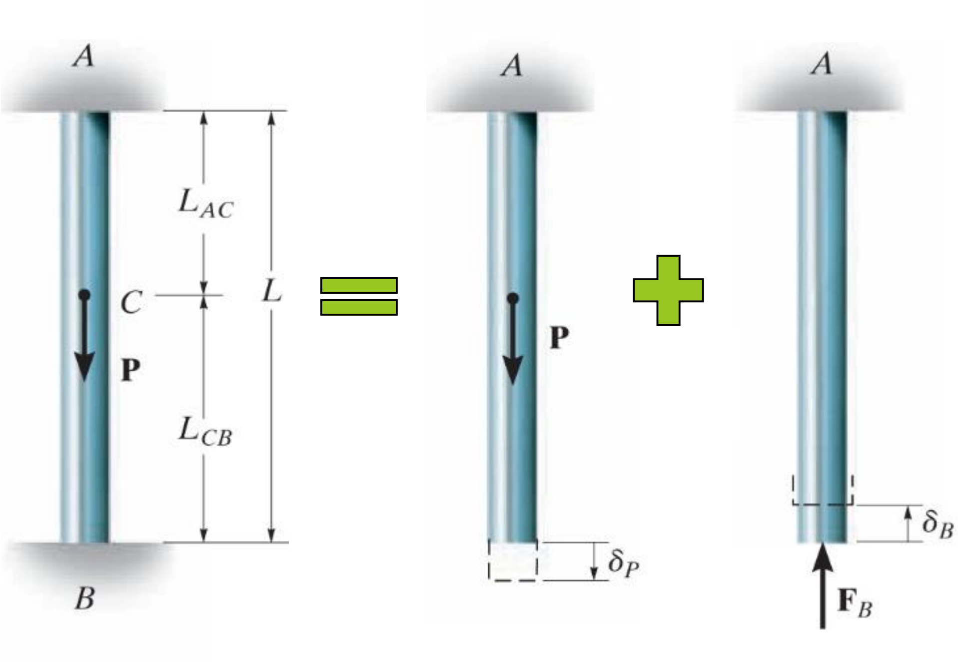
The Force Method for Axially Loaded Members

From the free-body diagram of the bar
The reaction at A can now be determined from the equation of equilibrium

$$+\uparrow \sum F = 0; \qquad F_A + F_B - P = 0$$

$$F_A + \frac{P.L_{AC}}{L} - P = 0$$

$$F_A = \frac{P.L_{CB}}{L}$$



Thermal Stress

A change in temperature can cause a body to change its dimensions. Generally, if the temperature increases, the body will expand, whereas if the temperature decreases, it will contract. Ordinarily this expansion or contraction is linearly related to the temperature increase or decrease that occurs

$$\delta_T = \alpha . \Delta T . L$$

Thermal Stress

If this is the case, and the material is homogeneous and isotropic; it has been found from experiment that the displacement of a member having a length L can be calculated using the formula:

$$\delta_T = \alpha . \Delta T \, . L$$

Where:

α	a property of the material, referred to as the linear coefficient of thermal expansion. The units measure strain per degree of temperature.
ΔT	the algebraic change in temperature of the member
L	the original length of the member
δ_T	the algebraic change in the length of the member

Thermal Stress

The change in length of a statically determinate member can easily be calculated using the equation:

$$\delta_T = \alpha . \Delta T . L$$

However, in a statically indeterminate member, these thermal displacements will be constrained by the supports, thereby producing thermal stresses that must be considered in design. Determining these thermal stresses is possible using the methods outlined in the previous sections

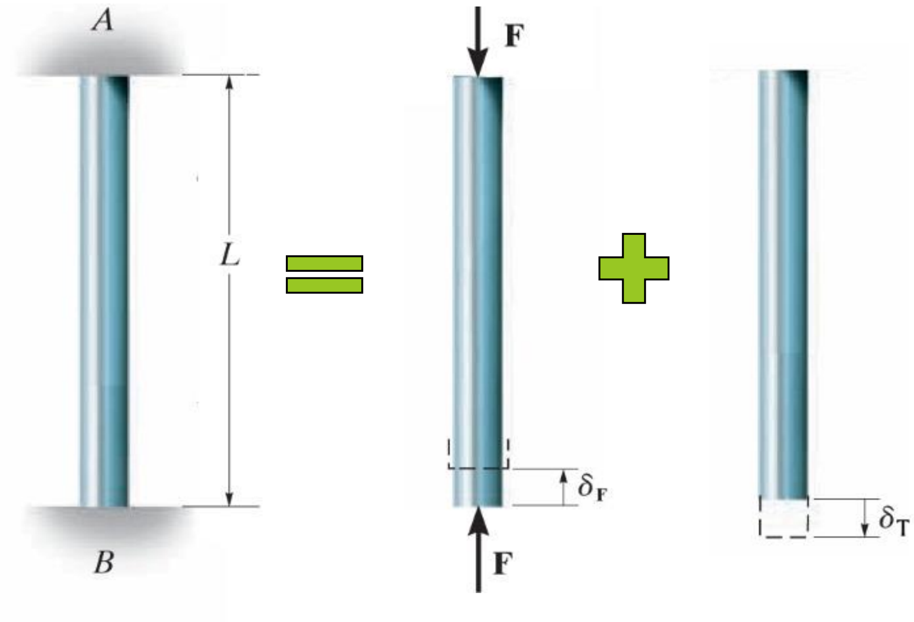
Thermal Stress

(1) = (2) + (3)

(1) No displacement at B

(2) Displacement at B when redundant force at B is applied

(3) Displacement at B when only the Thermal stress is applied

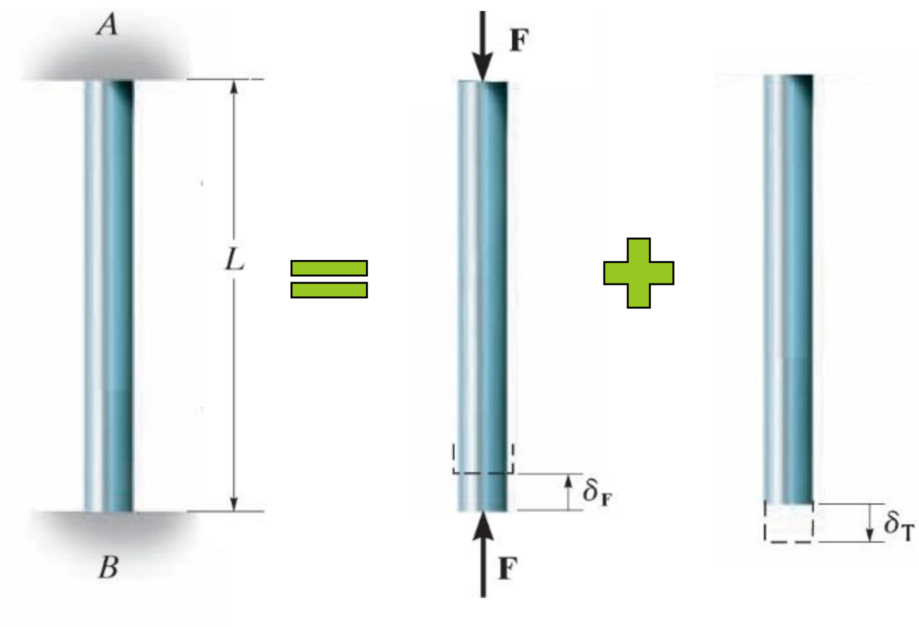


Thermal Stress

From the free-body diagram of the bar
The reaction at *A* can now be determined from the equation of equilibrium

$$+\uparrow \sum F = 0; \qquad F_A - F_B = 0$$

$$F_A = F_B = F$$



Thermal Stress

The compatibility equation for displacements at point B, for which we have assumed that displacements are positive downward.

$$\delta_T - \delta_B = 0$$

Applying the load–displacement relationship and thermal expansion

$$\alpha.\Delta T.L - \frac{F.L}{E.A} = 0$$

$$\alpha.\Delta T.E.A = F$$