Lecture 11

Stiffness Method

For beam analysis

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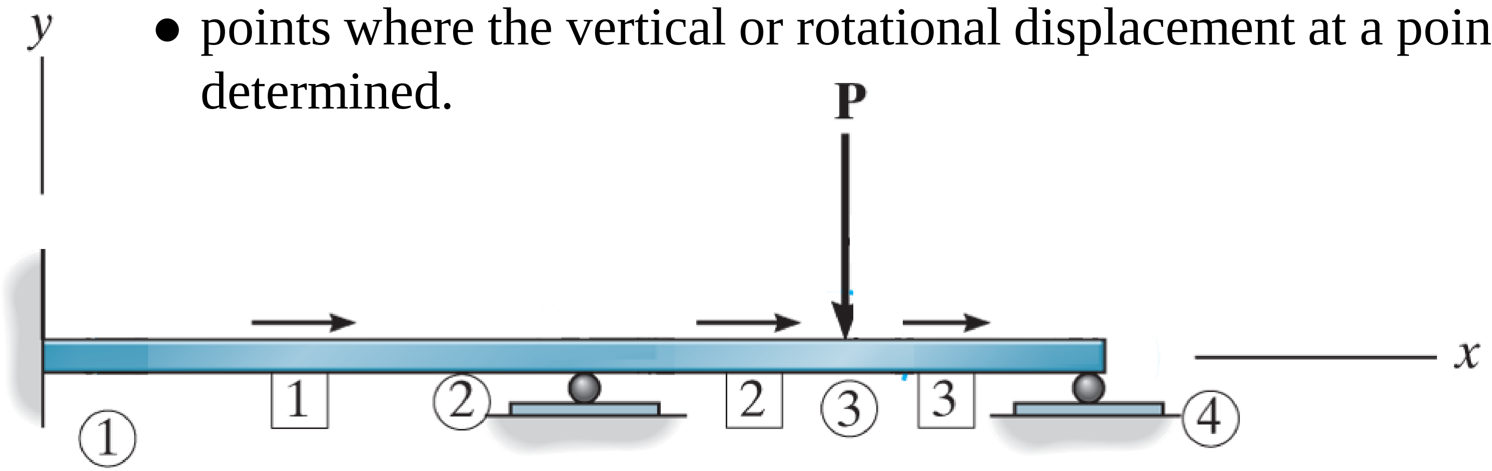
Preliminary remarks

Member & node identification

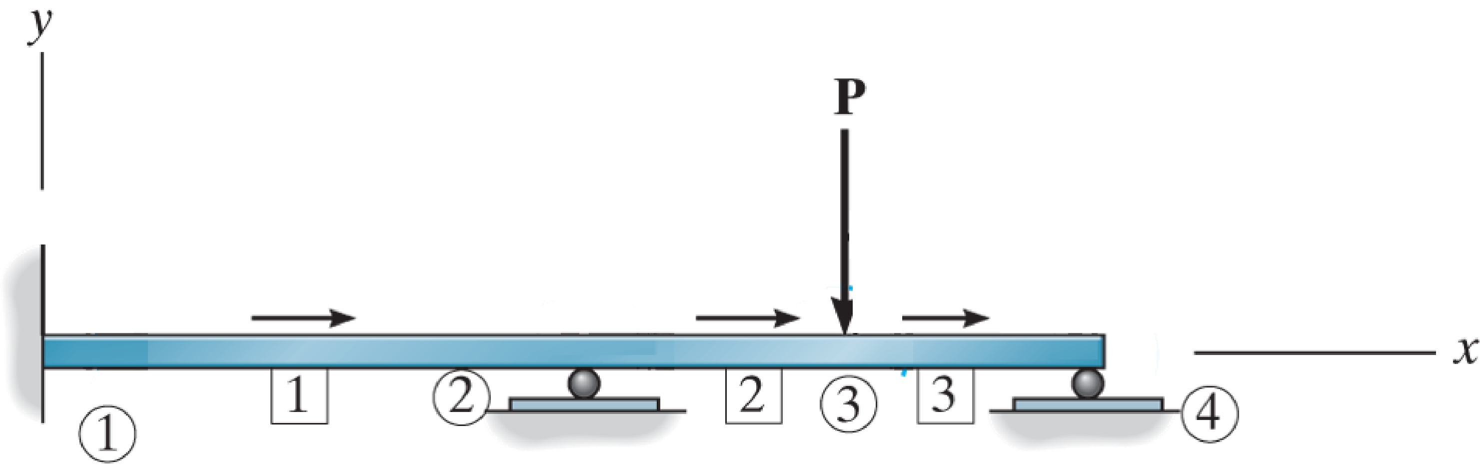
- The nodes of each element are located at :
  - Supports;
  - points where members are connected together;
  - points where an external force is applied;
  - points where the cross-sectional area suddenly changes;
  - points where the vertical or rotational displacement at a point is to be determined.



Preliminary remarks

Global & member coordinates

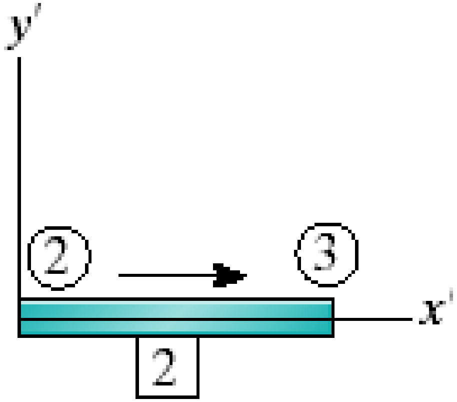
- The global coordinate system will be identified using x, y, z axes that generally have their origin at a node & are positioned so that the nodes at other points on the beam are positive coordinates.



Preliminary remarks

Global & member coordinates

- The local or member x’, y’, z’ coordinates have their origin at the near end of each element
- The positive x’ axis is directed towards the far end

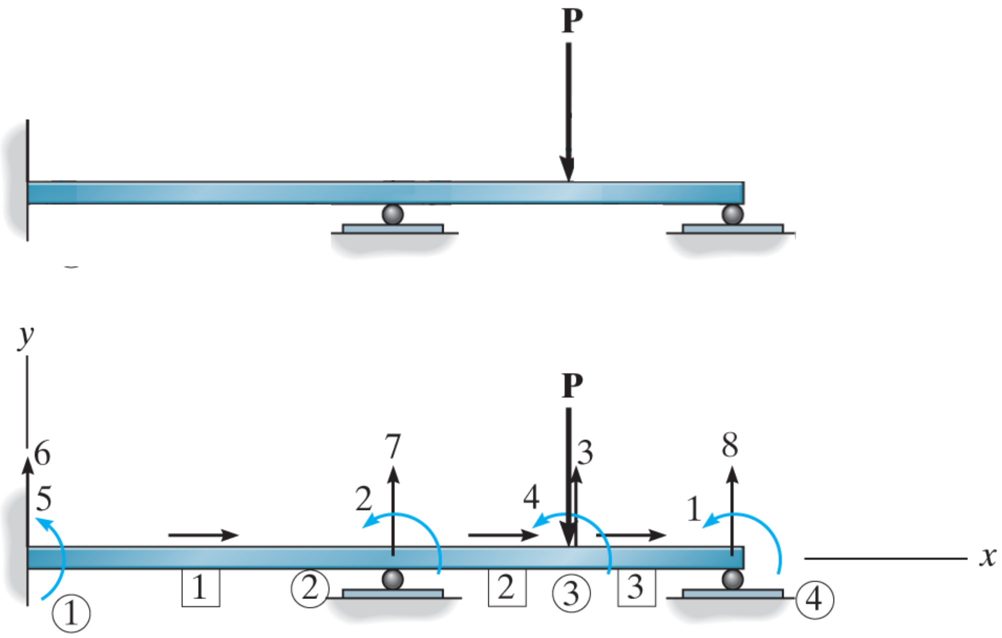


Preliminary remarks

Kinematic indeterminacy

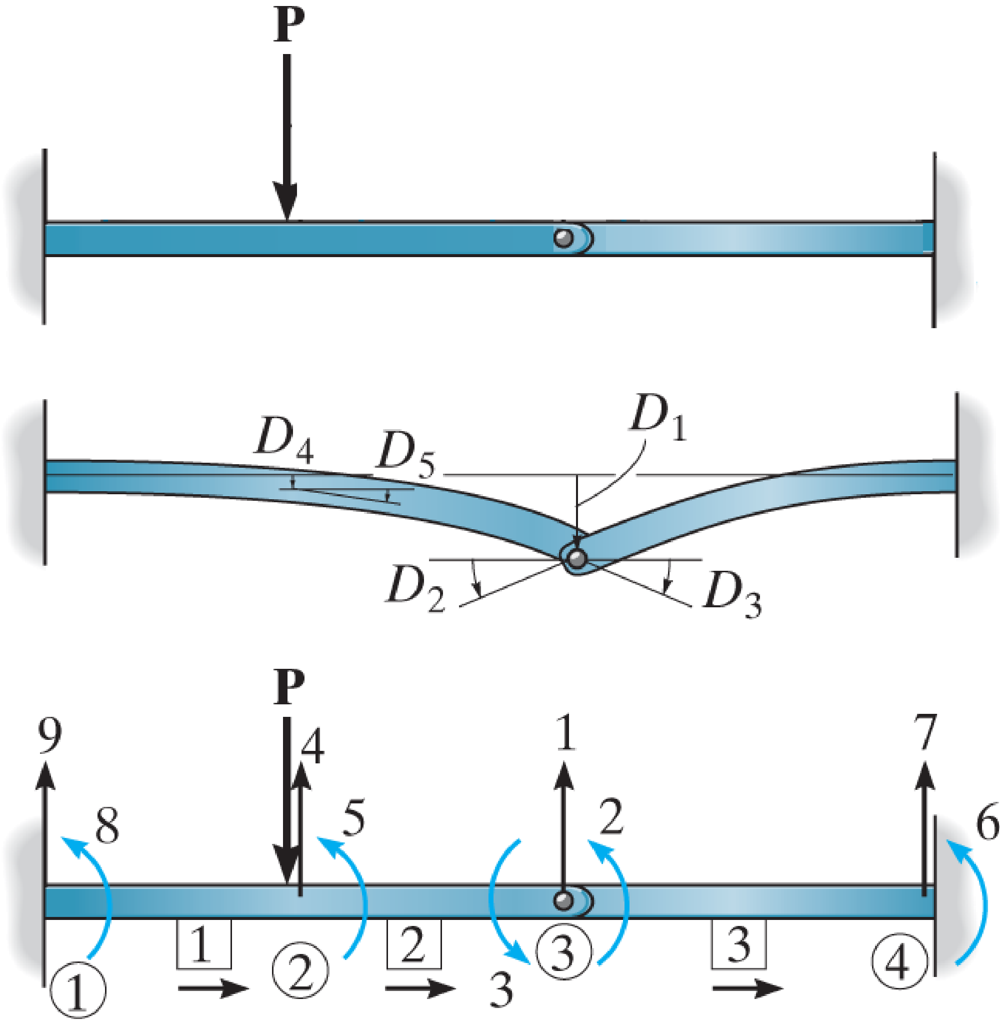
- The consider the effects of both bending & shear
- Each node on the beam can have 2 degrees of freedom:
  - Vertical displacement &
  - Rotation
- These displacement will be identified by code numbers
- The lowest code numbers will be used to identify the unknown displacement (unconstrained degrees of freedom) & the highest numbers are used to identify the known displacement (constrained degrees of freedom).

- Kinematic indeterminacy
  - The beam is kinematically indeterminate to the 4th degree
  - There are 8 degrees of freedom for which
    - code numbers 1 to 4 (unknown) displacement
    - code numbers 5 to 8 (known) displacement



Preliminary remarks

- Kinematic indeterminacy
  - The beam is kinematically indeterminate to the 5th degree
  - There are 9 degrees of freedom for which
    - code numbers 1 to 5 (unknown) displacement
    - code numbers 6 to 9 (known) displacement



Preliminary remarks

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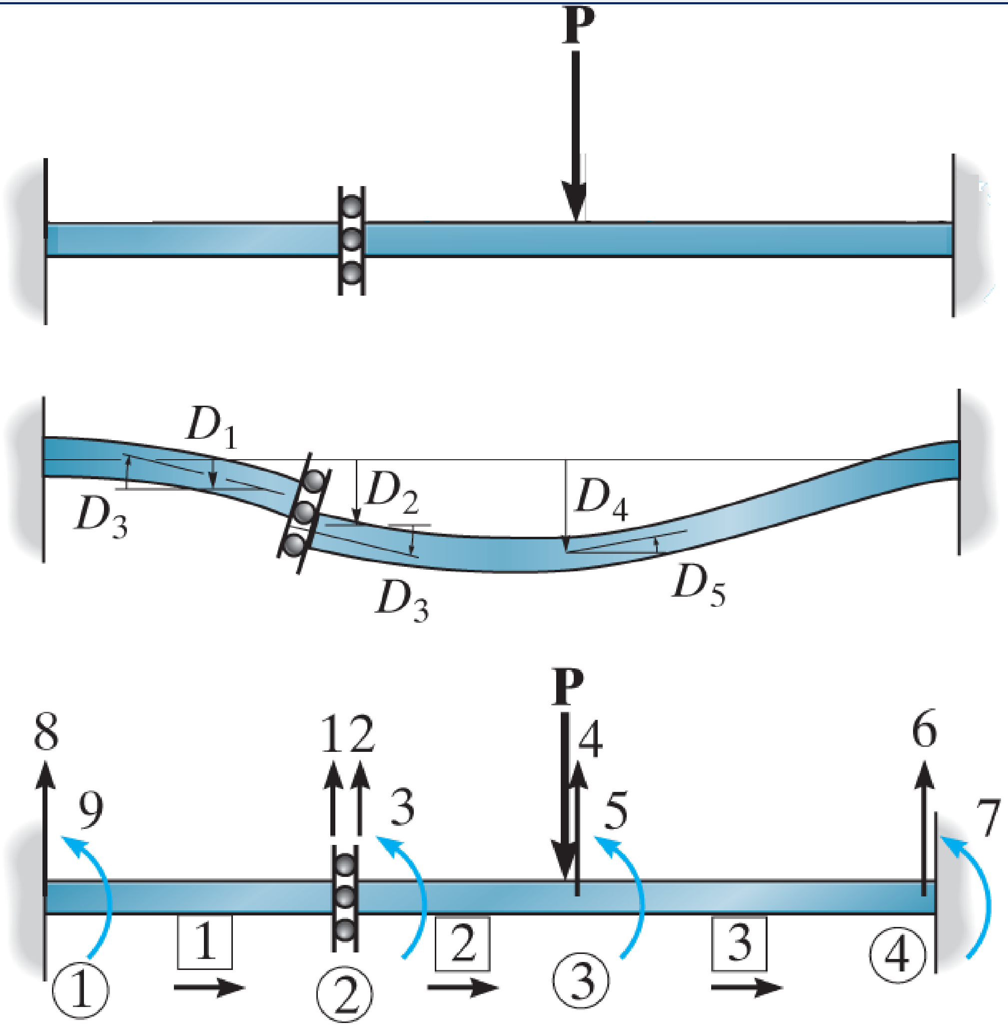
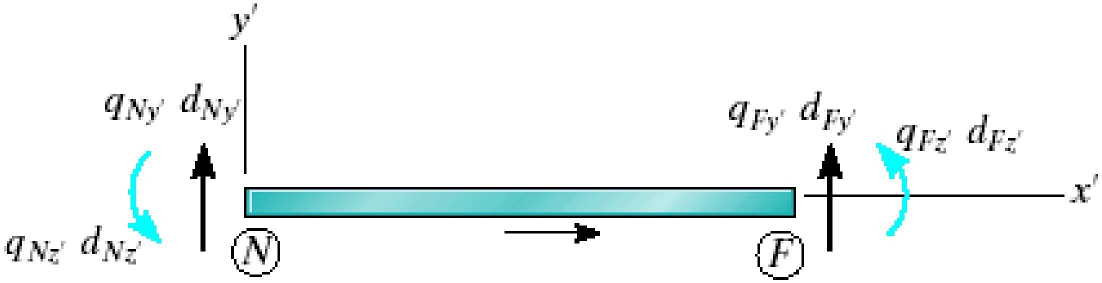


Preliminary remarks

- Kinematic indeterminacy
  - First we must establish the stiffness matrix for each element
  - These matrices are combined to form the beam or structure stiffness matrix
  - We can then proceed to determine the unknown displacement at the nodes
  - This will determine the reactions at the beam & the internal shear & moments at the nodes

Beam-member stiffness matrix

- We will develop the stiffness matrix for a beam element or member having a constant cross-sectional area & referenced from the local x’, y’, z’ coordinate system



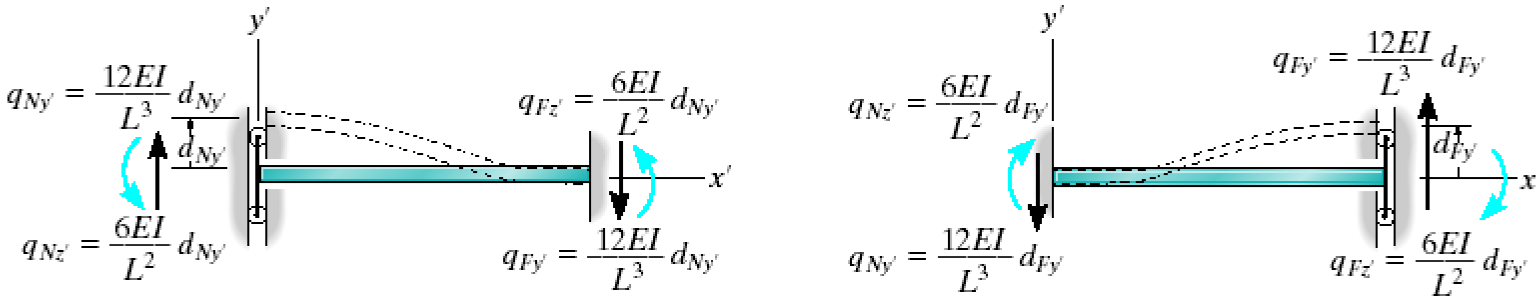
positive sign convention

Beam-member stiffness matrix

- Linear & angular displacement associated with these loadings also follow the same sign convention of the shear forces and bending moments.
- We will impose each of these displacement separately & then determine the loadings acting on the member caused by each displacement

Beam-member stiffness matrix

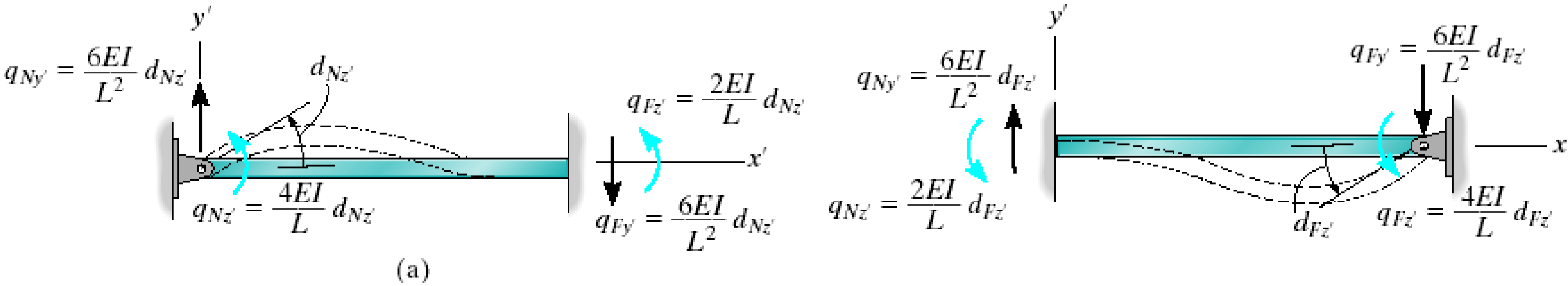
- y’ displacement
  - A positive displacement dNy’ is imposed while other possible displacement are prevented
  - The resulting shear forces & bending moments that are created are shown



y' displacements

Beam-member stiffness matrix

- z’ rotation
  - A positive rotation dNz’ is imposed while other possible displacement are prevented
  - The required shear forces & bending moments necessary for the deformation are shown
  - When dNz’ is imposed, the resultant loadings are shown



Beam-member stiffness matrix

- By superposition, the resulting four load-displacement relations for the member can be expressed in matrix form as

$$\begin{bmatrix} q_{Ny'} \\ q_{Nz'} \\ q_{Fy'} \\ q_{Fz'} \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} N_{y'} & N_{z'} & F_{y'} & F_{z'} \end{array} \\ \begin{bmatrix} \dfrac{12EI}{L^3} & \dfrac{6EI}{L^2} & -\dfrac{12EI}{L^3} & \dfrac{6EI}{L^2} \\ \dfrac{6EI}{L^2} & \dfrac{4EI}{L} & -\dfrac{6EI}{L^2} & \dfrac{2EI}{L} \\ -\dfrac{12EI}{L^3} & -\dfrac{6EI}{L^2} & \dfrac{12EI}{L^3} & -\dfrac{6EI}{L^2} \\ \dfrac{6EI}{L^2} & \dfrac{2EI}{L} & -\dfrac{6EI}{L^2} & \dfrac{4EI}{L} \end{bmatrix} \end{array} \begin{bmatrix} d_{Ny'} \\ d_{Nz'} \\ d_{Fy'} \\ d_{Fz'} \end{bmatrix}$$

- These equation can be written as

$$q = kd$$

Beam-structure stiffness matrix

- Once all the member stiffness matrices have been found, we must assemble them into the structure stiffness matrix, K
- The rows & columns of each k matrix, the equations are identified by the 2 code numbers at the near end (Ny’, Nz’) of the member followed by those at the far end (Fy’, Fz’)
- When assembling the matrices, each element must be placed in the same location of the K matrix

Application of the stiffness method for beam analysis

- K will have an order that will be equal to the highest code number assigned to the beam
- Where several members are connected at a node, their member stiffness influence coefficients will have the same position in the K matrix & therefore must be algebraically added to determine the nodal stiffness influence coefficient for the structure

Application of the stiffness method for beam analysis

- Once the stiffness matrix is determined, the loads at the nodes of the beam can be related to the displacement using the structure stiffness equation

$$Q = KD$$

- Partitioning the stiffness matrix into the known & unknown elements of load & disp, we have

$$\left[\begin{array}{c} \mathbf{Q}_k \\ \hdashline \mathbf{Q}_u \end{array}\right] = \left[\begin{array}{c:c} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \hdashline \mathbf{K}_{21} & \mathbf{K}_{22} \end{array}\right] \left[\begin{array}{c} \mathbf{D}_u \\ \hdashline \mathbf{D}_k \end{array}\right]$$

Application of the stiffness method for beam analysis

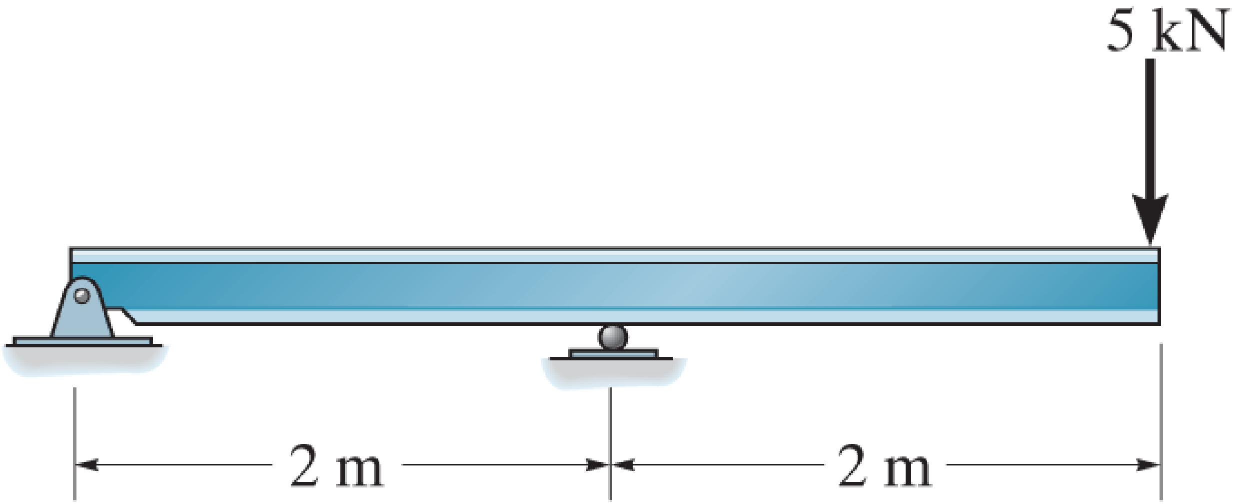
- This expands into 2 equation:

$$Q_k = K_{11}D_u + K_{12}D_k$$
$$Q_u = K_{21}D_u + K_{22}D_k$$

- The unknown displacement Du are determined from the first of these equation
- Using these values, the support reactions Qu are computed for the second equation

Example

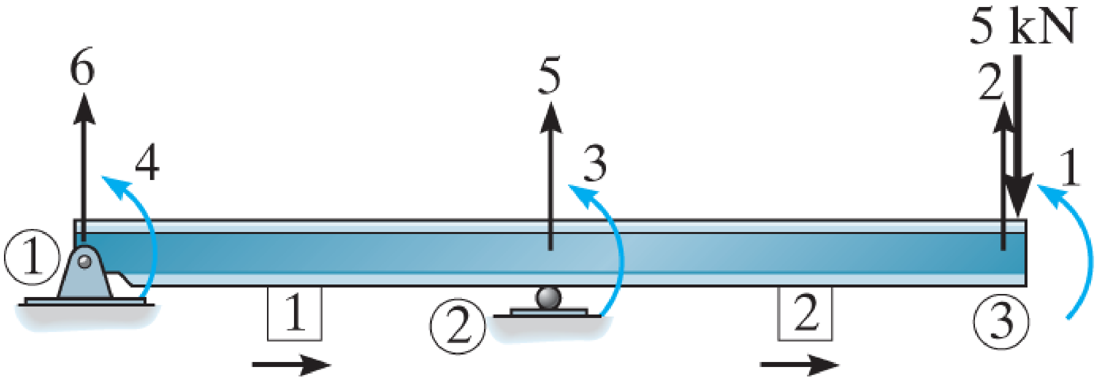
- Determine the reactions at the supports of the beam as shown. EI is constant.



Solution

- The beam has 2 elements & 3 nodes identified.
- The known load & displacement matrices are:

$$Q_k = \begin{array}{c} \begin{bmatrix} 0 \\ -5 \\ 0 \\ 0 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \qquad D_k = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\begin{array}{c} 5 \\ 6 \end{array}$$



- Each of the 2 member stiffness matrices can be determined.

$$\mathbf{k}_1 = EI \begin{array}{c} \begin{array}{cccc} 6 & 4 & 5 & 3 \end{array} \\ \left[\begin{array}{cccc} 1.5 & 1.5 & -1.5 & 1.5 \\ 1.5 & 2 & -1.5 & 1 \\ -1.5 & -1.5 & 1.5 & -1.5 \\ 1.5 & 1 & -1.5 & 2 \end{array}\right] \end{array} \begin{array}{c} \\ 6 \\ 4 \\ 5 \\ 3 \end{array}$$

$$\mathbf{k}_2 = EI \begin{array}{c} \begin{array}{cccc} 5 & 3 & 2 & 1 \end{array} \\ \left[\begin{array}{cccc} 1.5 & 1.5 & -1.5 & 1.5 \\ 1.5 & 2 & -1.5 & 1 \\ -1.5 & -1.5 & 1.5 & -1.5 \\ 1.5 & 1 & -1.5 & 2 \end{array}\right] \end{array} \begin{array}{c} \\ 5 \\ 3 \\ 2 \\ 1 \end{array}$$

Solution

- We can now assemble these elements into the structure stiffness matrix.
- The matrices are partitioned as shown,

$$\left[\begin{array}{c} 0 \\ -5 \\ 0 \\ 0 \\ \hline Q_5 \\ Q_6 \end{array}\right] = EI \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccc|cc} 2 & -1.5 & 1 & 0 & 1.5 & 0 \\ -1.5 & 1.5 & -1.5 & 0 & -1.5 & 0 \\ 1 & -1.5 & 4 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 2 & -1.5 & 1.5 \\ \hline 1.5 & -1.5 & 0 & -1.5 & 3 & -1.5 \\ 0 & 0 & 1.5 & 1.5 & -1.5 & 1.5 \end{array}\right] \end{array} \left[\begin{array}{c} D_1 \\ D_2 \\ D_3 \\ D_4 \\ \hline 0 \\ 0 \end{array}\right]$$

Solution

- Carrying out the multiplication for the first 4 rows, we have

$$0 = 2D_1 - 1.5D_2 + D_3 + 0$$

$$-\frac{5}{EI} = -1.5D_1 + 1.5D_2 - 1.5D_3 + 0$$

$$0 = D_1 - 1.5D_2 + 4D_3 + D_4$$

$$0 = 0 + 0 + D_3 + 2D_4$$

Solution

- Solving, we have:

$$D_1 = -\frac{16.67}{EI}, \quad D_2 = -\frac{26.67}{EI}$$

$$D_3 = -\frac{6.67}{EI}, \quad D_4 = \frac{3.33}{EI}$$

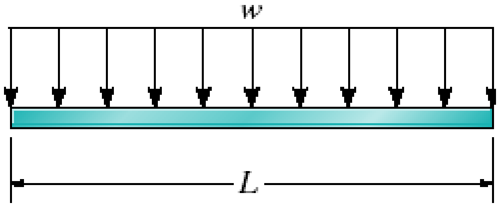
- Using these results, we get:

$$Q_5 = 10kN$$

$$Q_6 = -5kN$$

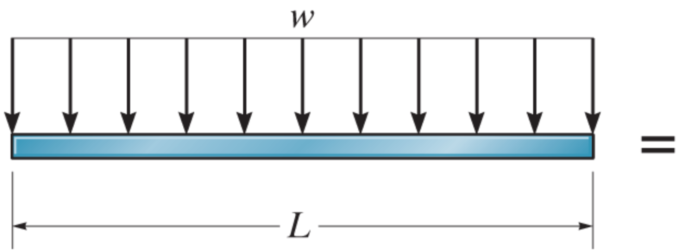
Application of the stiffness method for beam analysis

- Intermediate loadings
  - It is important that the elements of the beam be free of loading along its length
  - This is necessary as the stiffness matrix for each element was developed for loadings applied only at its ends
  - Consider the beam element of length L which is subjected to uniform distributed load, w



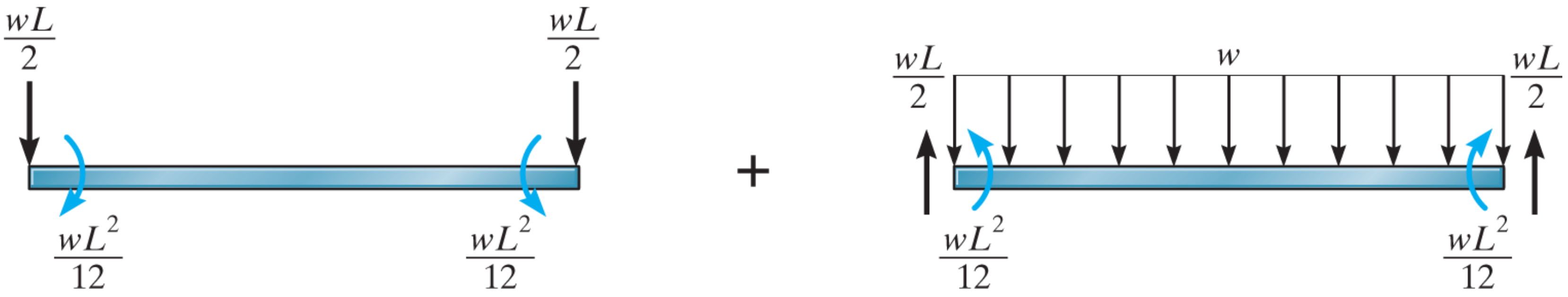
actual loading

Application of the stiffness method for beam analysis



actual loading

(a)



fixed-end
element loading
on joints

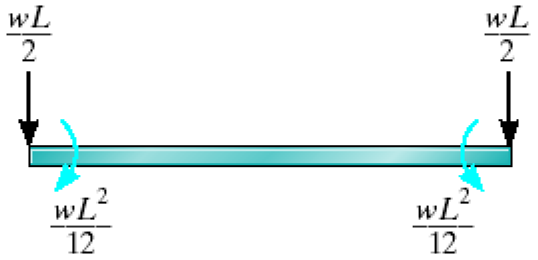
(b)

actual loading and
reactions on fixed-
supported element

(c)

Application of the stiffness method for beam analysis

- Intermediate loadings
  - First, we will apply fixed-end moments & reactions to the element which will be used in the stiffness method
  - We will refer to these loadings as a column matrix qo
  - The distributed loading within the beam is determined by adding these 2 results

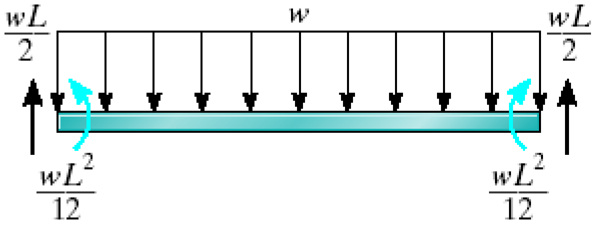


fixed-end
element loading
on joints

Application of the stiffness method for beam analysis

- Member forces
  - The shear & moment at the ends of each beam element can be determined by adding on any fixed end reactions qo if the element is subjected to an intermediate loading
  - We have:

$$\boxed{q = kd + q_0}$$



actual loading and reactions on fixed-supported element