Lecture 9

Stiffness Method (1)

For truss analysis

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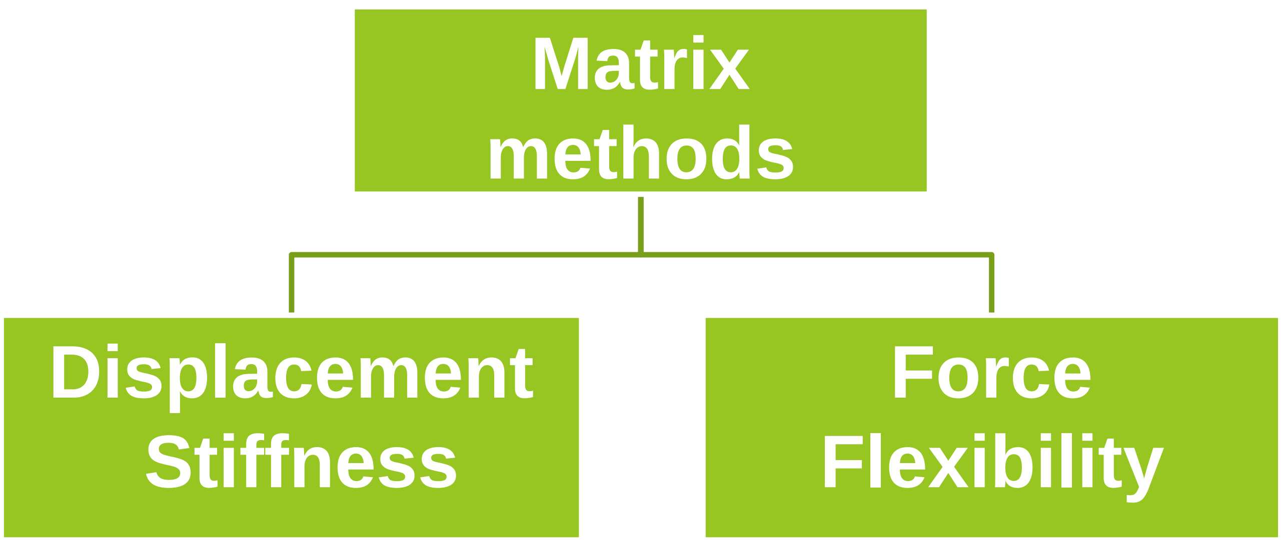
Matrix Methods

A structural problem may be formulated in either of two different ways. One approach proceeds with the displacements of the structure as the unknowns, the internal forces then follow from the determination of these displacements, while in the alternative approach, forces are treated as being initially unknown.



Matrix Methods

Stiffness method

- Displacement method of analysis
- Used for statically determinate and indeterminate structures
- Yields the displacements and forces directly.

Flexibility method

- Force method of analysis
- Requires a different procedure for each of these two cases
- The displacements are not obtained directly

Stiffness Method

The stiffness method:

- Is a displacement method of analysis
- Can be used to analyze both statically determinate and indeterminate structures
- Yields the displacement & forces directly
- It is generally easy to formulate the necessary matrices for the computer using the stiffness method, and once this is done, the computer calculations can be performed efficiently.

Stiffness Method

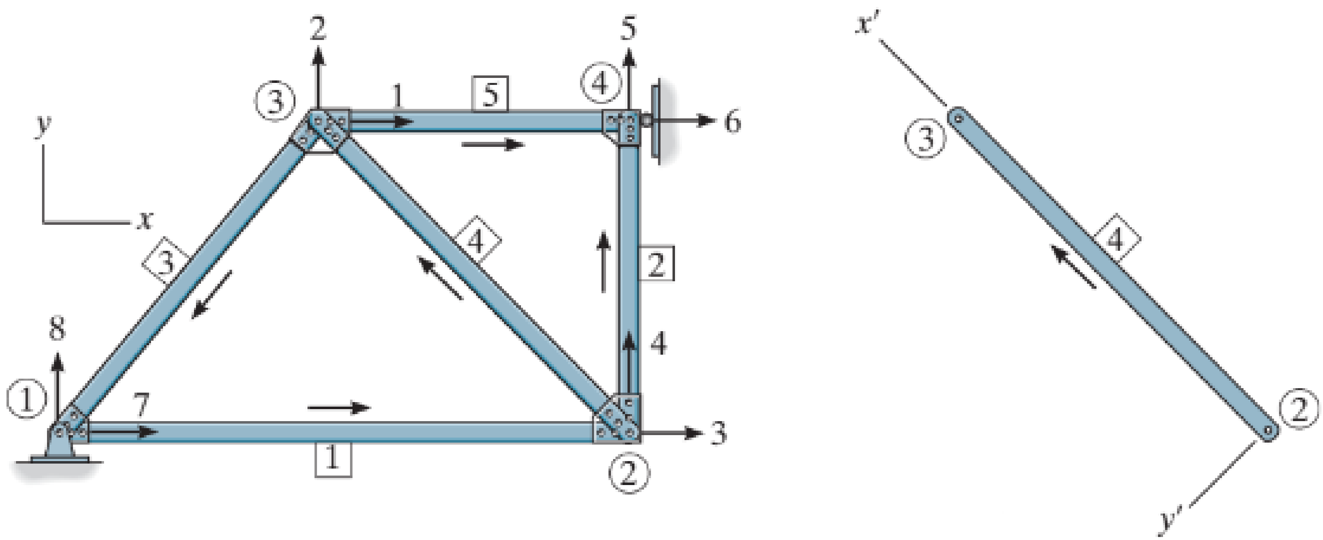
- Application of the stiffness method requires subdividing the structure into a series of discrete finite elements & identifying their end points as nodes
- For truss analysis, the finite elements are represented by each of the members that compose the truss & the nodes represent the joints
- The force-displacement properties of each element are determined & then related to one another using the force equilibrium equations written at the nodes

Stiffness Method

- These relationships for the entire structure are then grouped together into the structure stiffness matrix, K
- The unknown displacement of the nodes can then be determined for any given loading on the structure
- When these displacements are known, the external & internal forces in the structure can be calculated using the force-displacement relations for each member

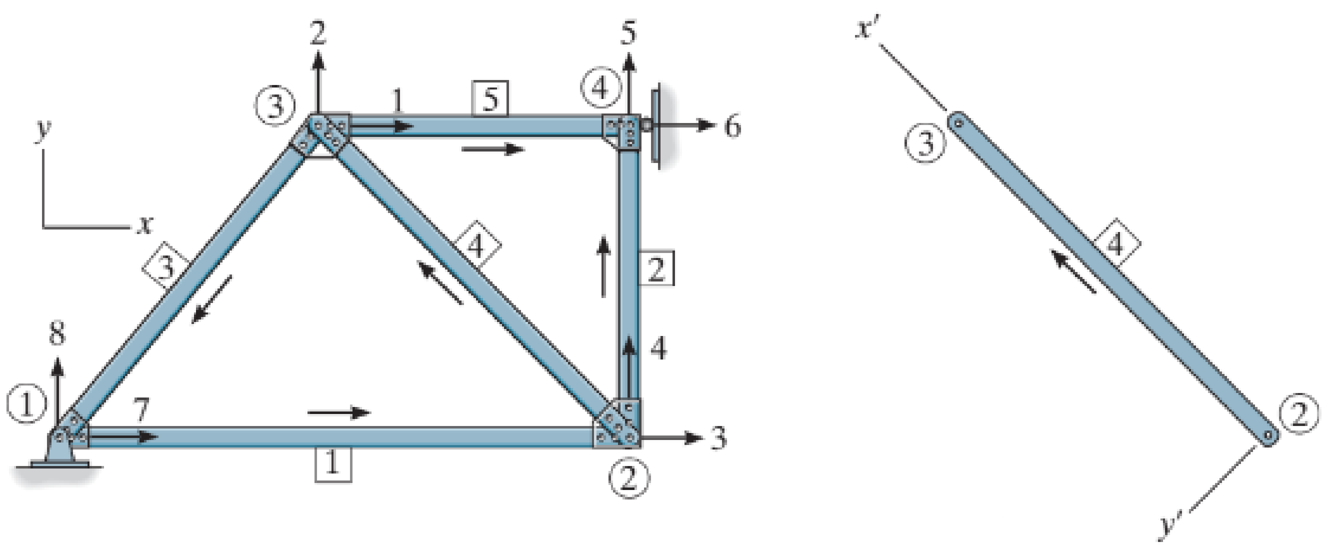
Preliminary definitions and concepts.

✓ Elements: to identify the elements or members of the structure, we will specify each member by a number enclosed within a square.



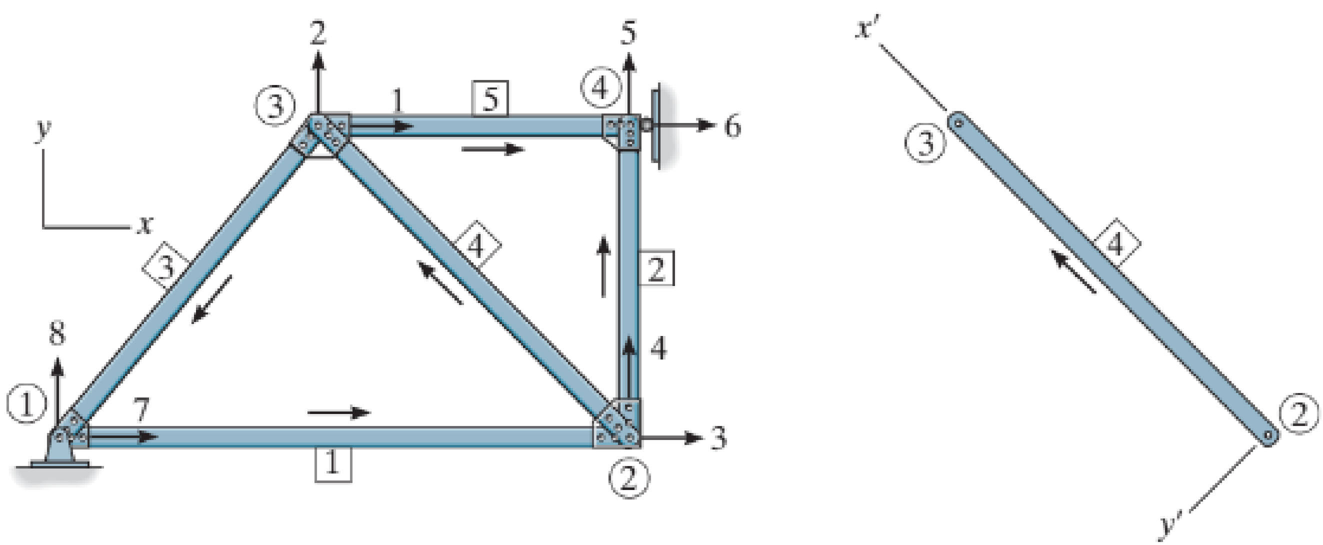
Preliminary definitions and concepts.

✓ Nodes: the Nodes are usually located at joints, supports, at the ends of a member, or where the members have a sudden change in cross section. To identify the elements or members nodes, We will specify each member by a number enclosed within a circle.



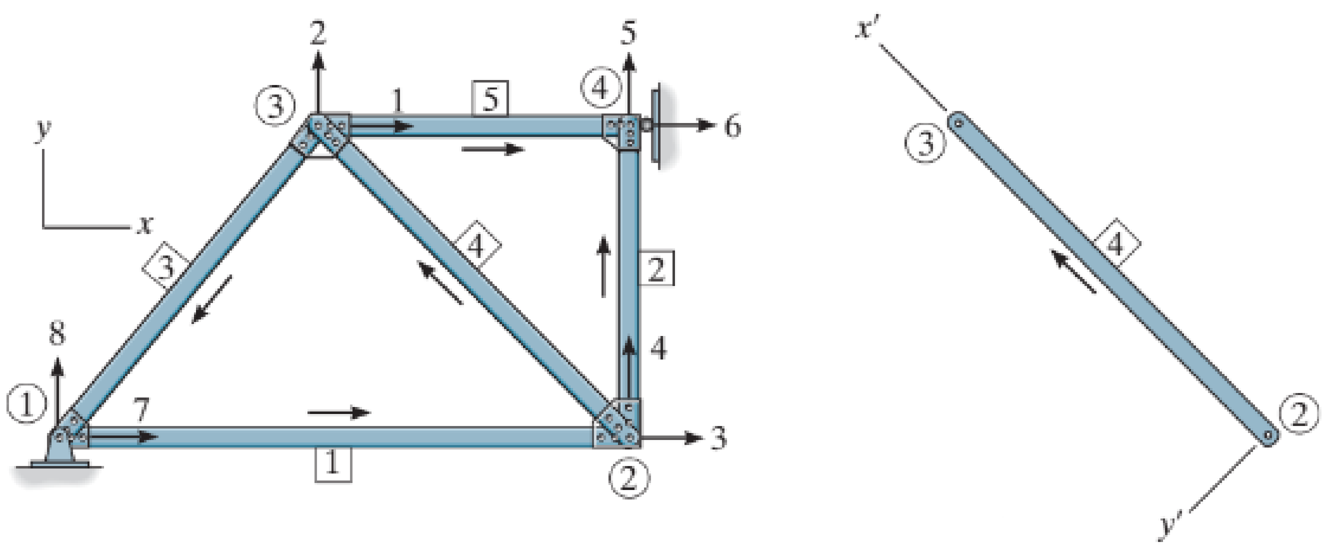
Preliminary definitions and concepts.

✓ Direction: to identify the "near" and "far" ends of the member, This will be done using an arrow written along the member, with the head of the arrow directed toward the far end.



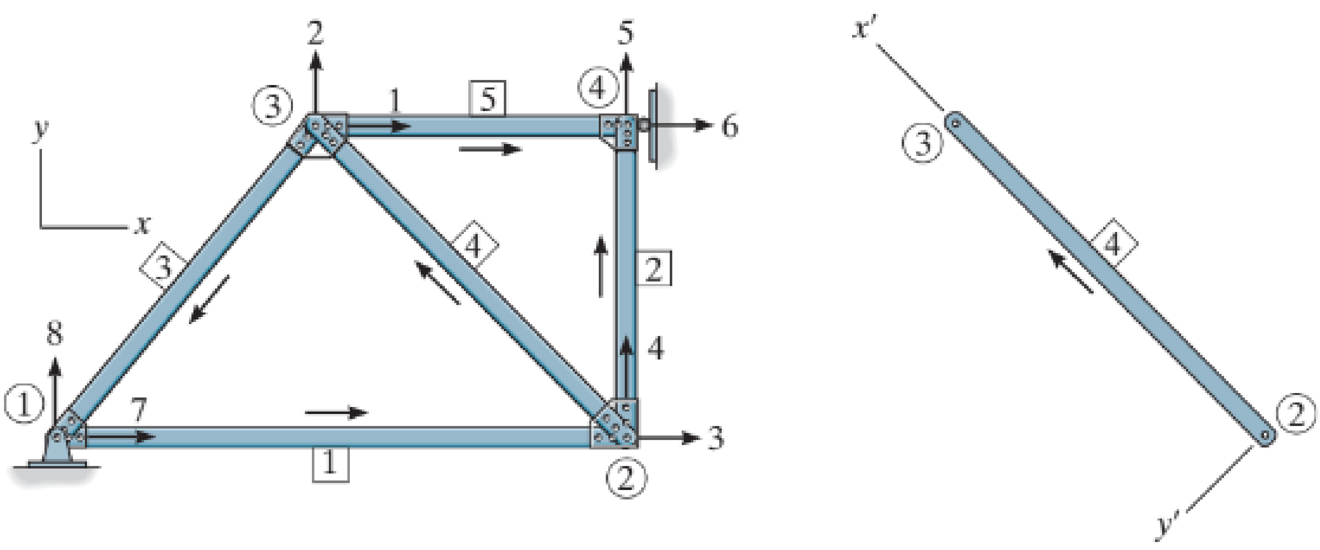
Preliminary definitions and concepts.

✓ Global or structure coordinate system: A single global or structure coordinate system, x, y, will be used to specify the sense of each of the external force and displacement components at the nodes.



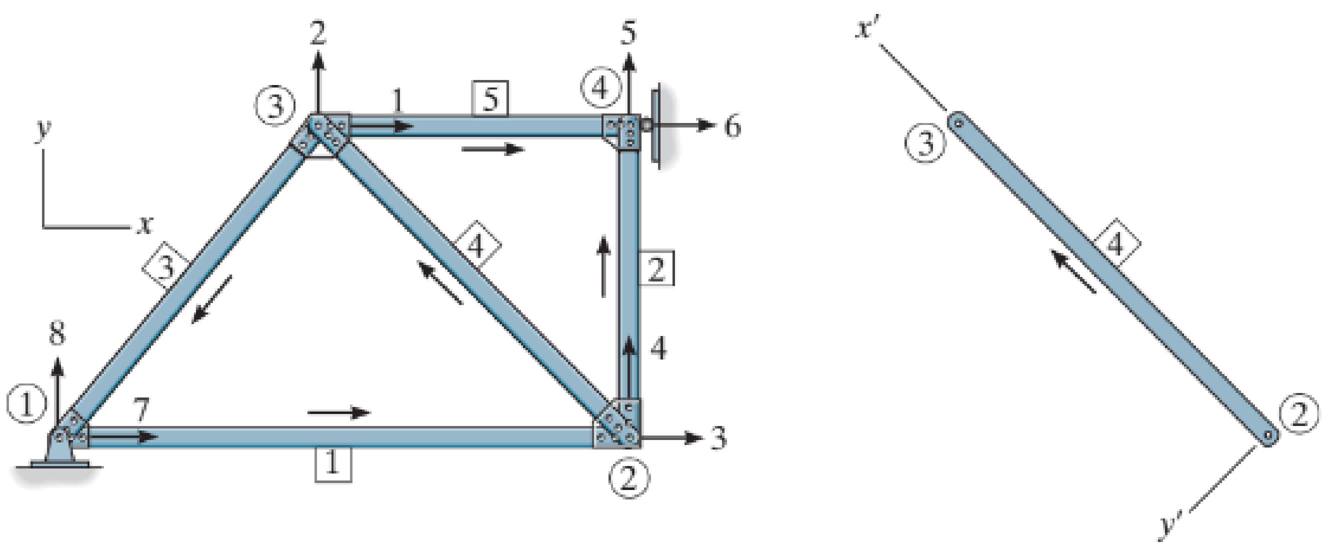
Preliminary definitions and concepts.

- ✓ Local or member coordinate system: it will be used for each member to specify the sense of direction of its displacements and internal loadings. It is identified using x', y', axes with the origin at the "near" node and the x' axis extending toward the "far" node.



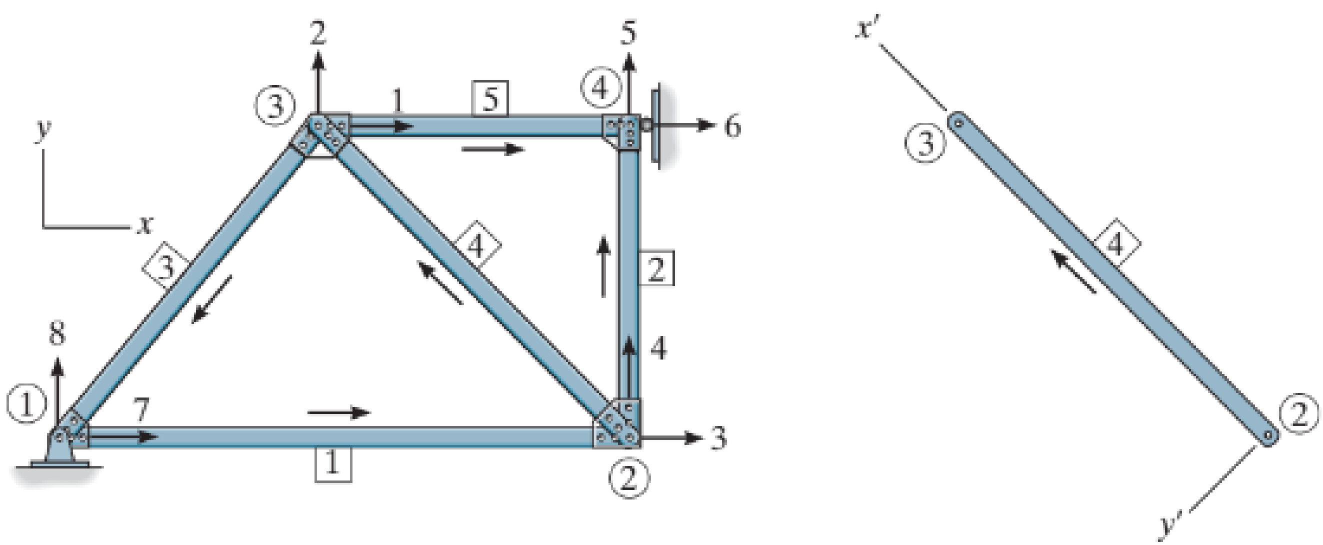
Preliminary definitions and concepts.

- ✓ **Degrees of Freedom:** for loaded structure, the nodes, will undergo unknown displacements, referred to as the degrees of freedom for the structure, for the displacement method it is important to specify these degrees of freedom since they become the unknowns when the method is applied.

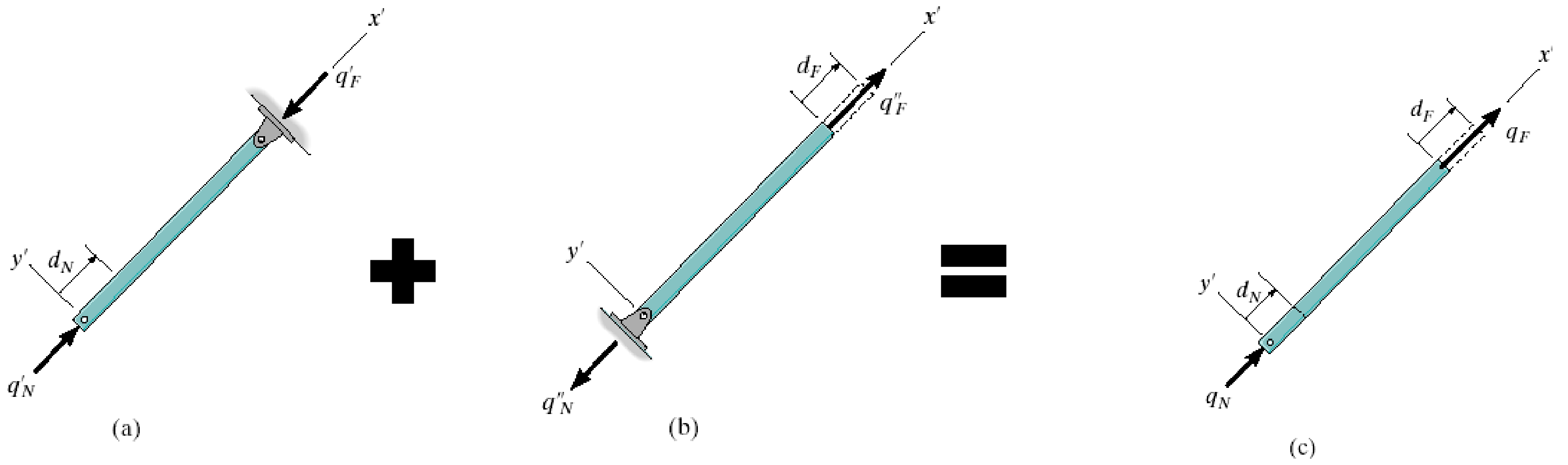


Preliminary definitions and concepts.

✓ Kinematic indeterminacy: The structure consists of a series of members connected to nodes, In 3D, each node on a frame or beam can have at most three linear displacements and three rotational displacements; and in 2D, each node can have at most two linear displacements and one rotational displacement.



Member stiffness matrix

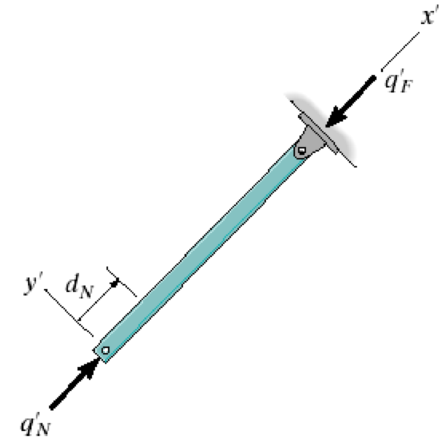


Member stiffness matrix

Establish the stiffness matrix for a single truss member using local *x'* and *y'* coordinates:

- ✓ A positive displacement d_N is imposed on the near end of the member while the far end is held pinned
- ✓ The forces developed at the ends of the members are:

$$q'_N = \frac{AE}{L} d_N; \quad q'_F = -\frac{AE}{L} d_N$$



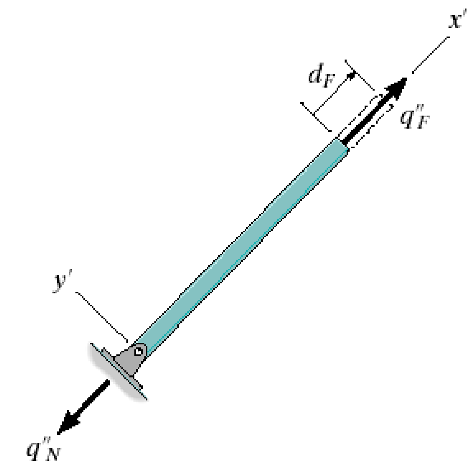
(a)

Member stiffness matrix

Establish the stiffness matrix for a single truss member using local *x*’ and *y*’ coordinates:

- ✓ Likewise, a positive displacement d_N is imposed on the far end of the member while the near end is held pinned
- ✓ The forces developed at the ends of the members are:

$$q''_N = -\frac{AE}{L} d_F; \quad q''_F = \frac{AE}{L} d_F$$

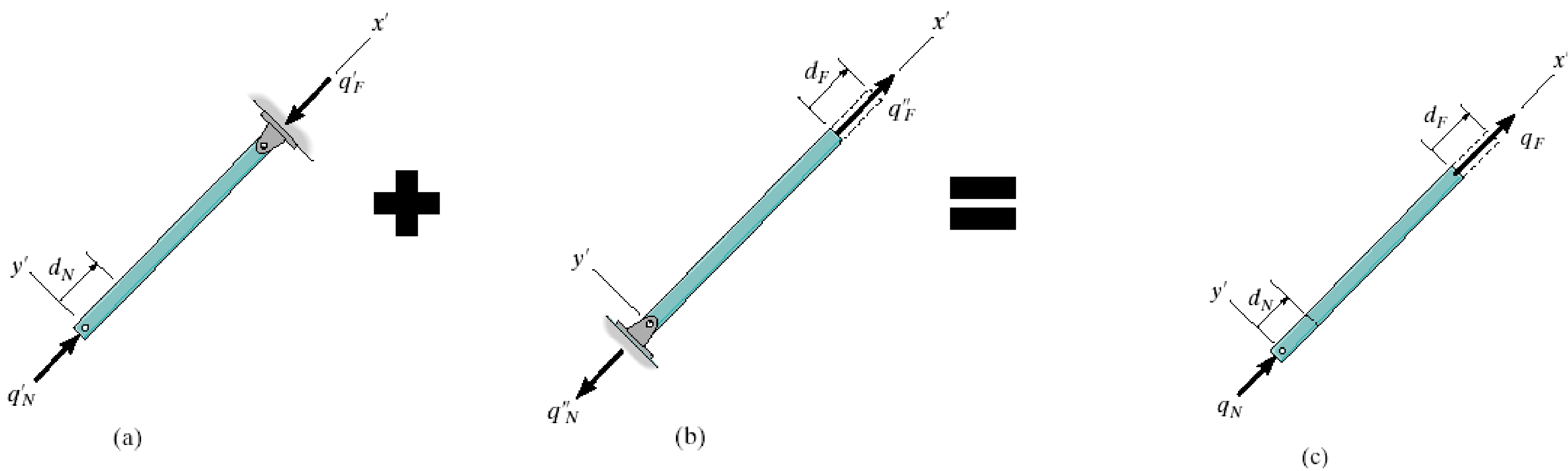


(b)

Member stiffness matrix

Establish the stiffness matrix for a single truss member using local x' and y' coordinates:

✓ By superposition, the resultant forces caused by both displacement are:



Member stiffness matrix

Establish the stiffness matrix for a single truss member using local *x*' and *y*' coordinates:

✓ By superposition, the resultant forces caused by both displacement are:

$$\left\{\begin{matrix} q'_N = \dfrac{AE}{L} d_N ; \quad q'_F = -\dfrac{AE}{L} d_N \\ + \\ q''_N = -\dfrac{AE}{L} d_F ; \quad q''_F = \dfrac{AE}{L} d_F \end{matrix}\right\} = \begin{matrix} q_N = \dfrac{AE}{L} d_N - \dfrac{AE}{L} d_F \\ q_F = \dfrac{AE}{L} d_F - \dfrac{AE}{L} d_N \end{matrix}$$

Member stiffness matrix

These load-displacement equations may be written in matrix form as:

$$\begin{bmatrix} q_N \\ q_F \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} d_N \\ d_F \end{bmatrix}$$

$$q = k'd$$

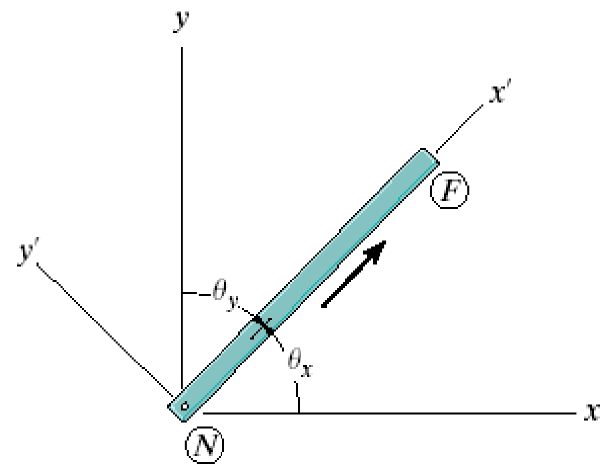
The matrix, k' is called the member stiffness matrix? and it is of the same form for each member of the truss. The four elements that comprise it are called member stiffness influence coefficients

$$k' = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Displacement & Force Transformation matrices

Since a truss is composed of many members, we will develop a method for transforming the member forces q and Displacement d defined in local coordinates to global coordinates

Global coordinates convention: positive x to the right and positive y upward θ_x and θ_y as shown

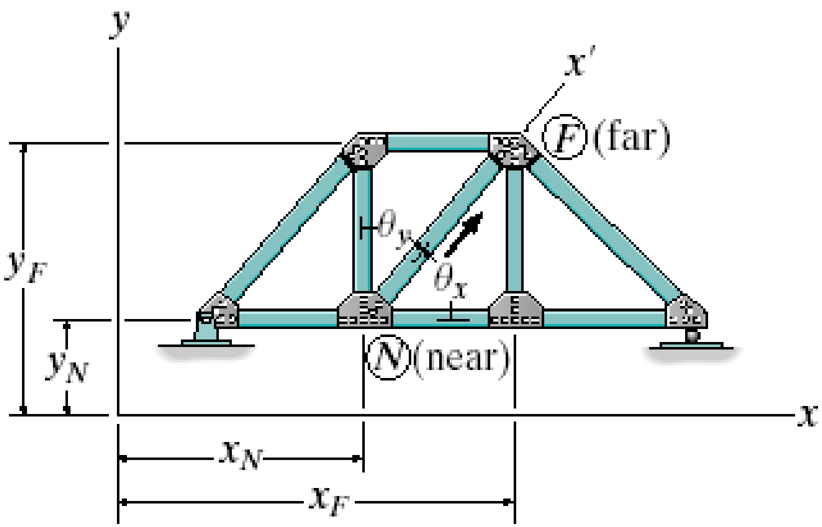


Displacement & Force Transformation matrices

- The cosines of these angles will be used in the matrix analysis as follows

$$\lambda_x = \cos\theta_x; \quad \lambda_y = \cos\theta_y$$

- For e.g. consider member NF of the truss as shown
- The coordinates of N & F are (x_N, y_N) and (x_F, y_F)



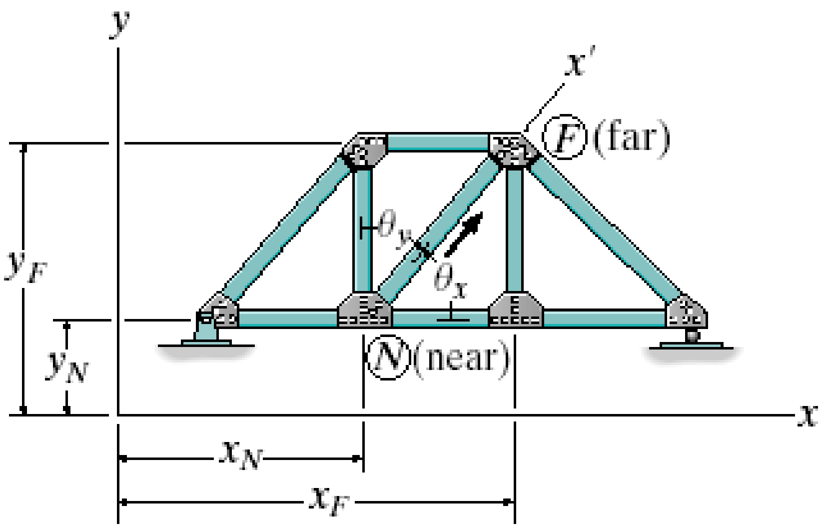
Member global stiffness matrix

$$\lambda_x = \cos\theta_x = \frac{x_F - x_N}{L}$$

$$= \frac{x_F - x_N}{\sqrt{(x_F - x_N)^2 + (y_F - y_N)^2}}$$

$$\lambda_y = \cos\theta_y = \frac{y_F - y_N}{L}$$

$$= \frac{y_F - y_N}{\sqrt{(x_F - x_N)^2 + (y_F - y_N)^2}}$$

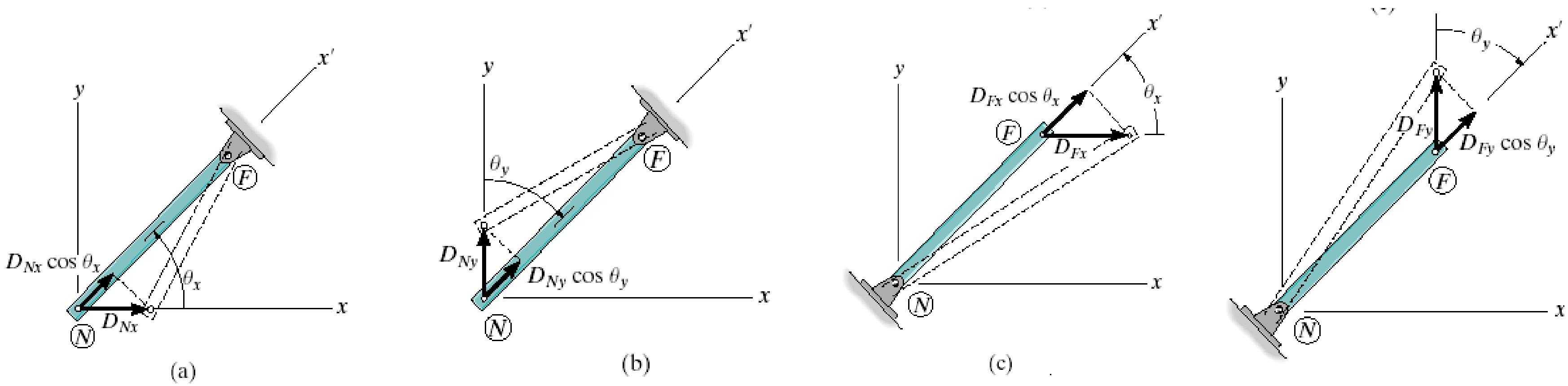


Displacement Transformation Matrix

In global coordinates each end of the member can have 2 degrees of freedom or independent Displacement; namely:

– joint N has D_{Nx} and D_{Ny}

– Joint F has D_{Fx} and D_{Fy}



Displacement Transformation Matrix

When the far end is held pinned & the near end is given a global displacement D_{Nx}, the corresponding displacement along member is $D_{Nx}.\cos\theta_x$

A displacement D_{Ny} will cause the member to be displaced $D_{Ny}.\cos\theta_y$ along the x' axis

$$d_N = D_{N_x}\cos\theta_x + D_{N_y}\cos\theta_y$$

$$d_F = D_{F_x}\cos\theta_x + D_{F_y}\cos\theta_y$$

Displacement Transformation matrix

$$\text{Let } \lambda_x = \cos\theta_x; \quad \lambda_y = \cos\theta_y$$

$$d_N = D_{N_x}\lambda_x + D_{N_y}\lambda_y; \quad d_F = D_{F_x}\lambda_x + D_{F_y}\lambda_y$$

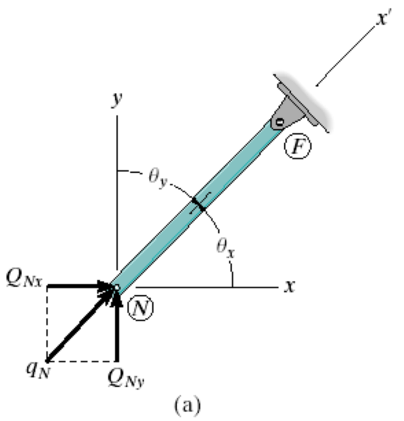
In matrix form,

$$\begin{bmatrix} d_N \\ d_F \end{bmatrix} = \begin{bmatrix} \lambda_x & \lambda_y & 0 & 0 \\ 0 & 0 & \lambda_x & \lambda_y \end{bmatrix} \begin{bmatrix} D_{N_x} \\ D_{N_y} \\ D_{F_x} \\ D_{F_y} \end{bmatrix}$$

$$d = TD$$

Force Transformation matrix

$$Q_{N_x} = q_N \cos\theta_x;\; Q_{N_y} = q_N \cos\theta_y$$

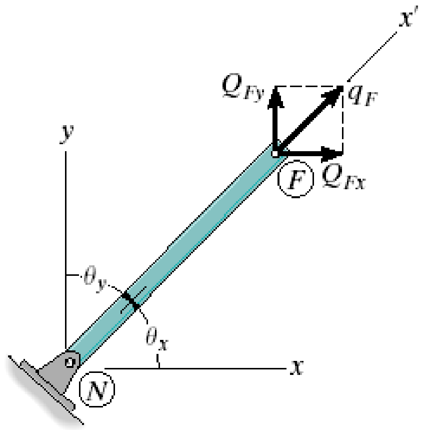


– If qF is applied to the bar,
  the global force components at F are:

– Using $Q_{F_x} = q_F \cos\theta_x;\; Q_{F_y} = q_F \cos\theta_y$

$$\lambda_x = \cos\theta_x; \quad \lambda_y = \cos\theta_y$$

$$Q_{N_x} = q_N \lambda_x;\; Q_{N_y} = q_N \lambda_y$$
$$Q_{F_x} = q_F \lambda_x;\; Q_{F_y} = q_F \lambda_y$$



Force Transformation matrix

– In matrix form

$$\begin{bmatrix} Q_{N_x} \\ Q_{N_y} \\ Q_{F_x} \\ Q_{F_y} \end{bmatrix} = \begin{bmatrix} \lambda_x & 0 \\ \lambda_y & 0 \\ 0 & \lambda_x \\ 0 & \lambda_y \end{bmatrix} \begin{bmatrix} q_N \\ q_F \end{bmatrix}$$

$$Q = T^T q$$

Force Transformation matrix

- In this case, T^T transforms the 2 local forces q acting at the ends of the member into 4 global force components Q
- This force transformation matrix is the transpose of the displacement transformation matrix

Member global stiffness matrix

✓ We can determine the member's forces q in terms of the global displacement D at its end points

$$\left.\begin{aligned} q &= k'd \\ d &= TD \end{aligned}\right\} \Rightarrow q = k'TD$$

✓ Substitution yields the final result:

$$\left.\begin{aligned} q &= k'TD \\ Q &= T^T q \end{aligned}\right\} \Rightarrow Q = \underbrace{T^T k'TD}_{k = T^T k'T} \Leftrightarrow Q = kD$$

Member global stiffness matrix

$$k = T^T k' T$$

$$T = \begin{bmatrix} \lambda_x & \lambda_y & 0 & 0 \\ 0 & 0 & \lambda_x & \lambda_y \end{bmatrix}$$

$$T^T = \begin{bmatrix} \lambda_x & 0 \\ \lambda_y & 0 \\ 0 & \lambda_x \\ 0 & \lambda_y \end{bmatrix}$$

$$k' = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{k} = \frac{AE}{L}\begin{array}{c} \begin{array}{cccc} N_x & N_y & F_x & F_y \end{array} \\ \begin{bmatrix} \lambda_x^2 & \lambda_x\lambda_y & -\lambda_x^2 & -\lambda_x\lambda_y \\ \lambda_x\lambda_y & \lambda_y^2 & -\lambda_x\lambda_y & -\lambda_y^2 \\ -\lambda_x^2 & -\lambda_x\lambda_y & \lambda_x^2 & \lambda_x\lambda_y \\ -\lambda_x\lambda_y & -\lambda_y^2 & \lambda_x\lambda_y & \lambda_y^2 \end{bmatrix} \end{array} \begin{array}{c} \\ N_x \\ N_y \\ F_x \\ F_y \end{array}$$

Truss stiffness matrix

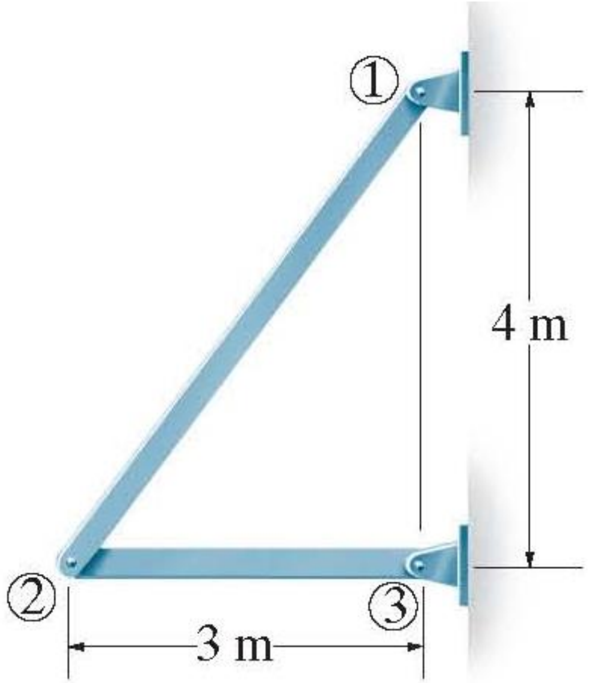
- ✓ Once all the member stiffness matrices are formed in the global coordinates, it becomes necessary to assemble them in the proper order so that the stiffness matrix K for the entire truss can be found
- ✓ This is done by designating the rows & columns of the matrix by the 4 code numbers used to identify the 2 global degrees of freedom that can occur at each end of the member

Truss stiffness matrix

- ✓ The structure stiffness matrix will then have an order that will be equal to the highest code number assigned to the truss since this rep the total no. of degree of freedom for the structure
- ✓ This method of assembling the member matrices to form the structure stiffness matrix will now be demonstrated by numerical e.g.
- ✓ This process is somewhat tedious when performed by hand but is rather easy to program on computer

Example 1

Determine the structure stiffness matrix for the 2 member truss as shown. AE is constant.



Example 1

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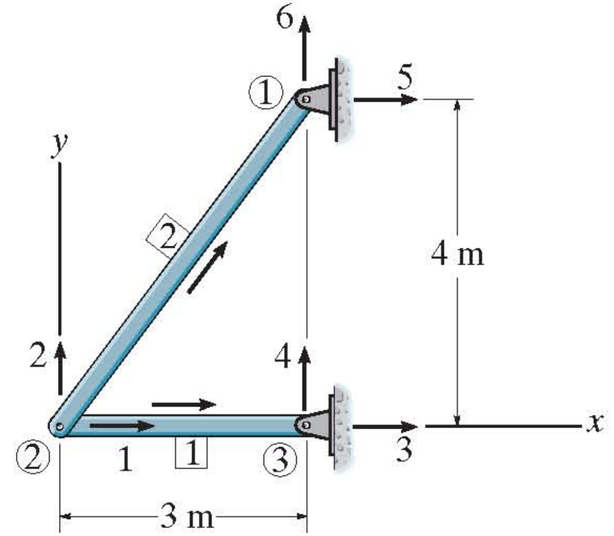


Example 1

By inspection, member 2 will have 2 unknown displacement components whereas joint 1 & 3 are constrained from disp.

Consequently, the displacement component at joint 2 are code numbered first, followed by those at joints 3 & 1.

The origin of the global coordinate system can be located at any point.

The members are identified arbitrarily & arrows are written along 2 members to identify the near & far ends of each member.

The direction cosines & the stiffness matrix for each member can now be determined.

Example 1

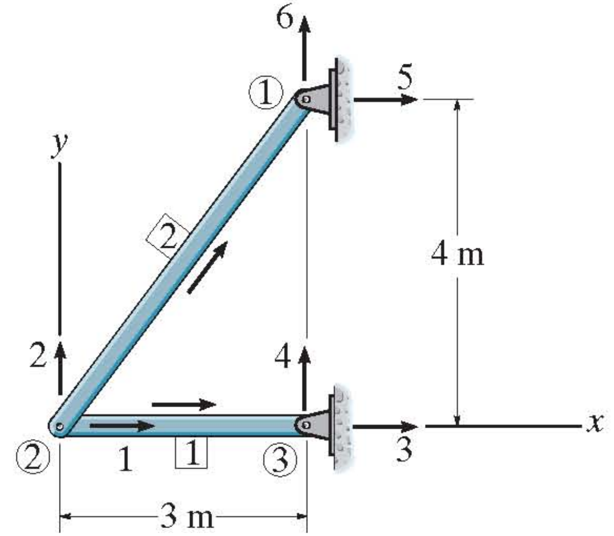
Member 1

- We have:

$$\lambda_x = \frac{3-0}{3} = 1; \quad \lambda_y = \frac{0-0}{3} = 0$$

- dividing each element by L = 3m, we have:

$$\mathbf{k}_1 = AE \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.333 & 0 & -0.333 & 0 \\ 0 & 0 & 0 & 0 \\ -0.333 & 0 & 0.333 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$



Example 1

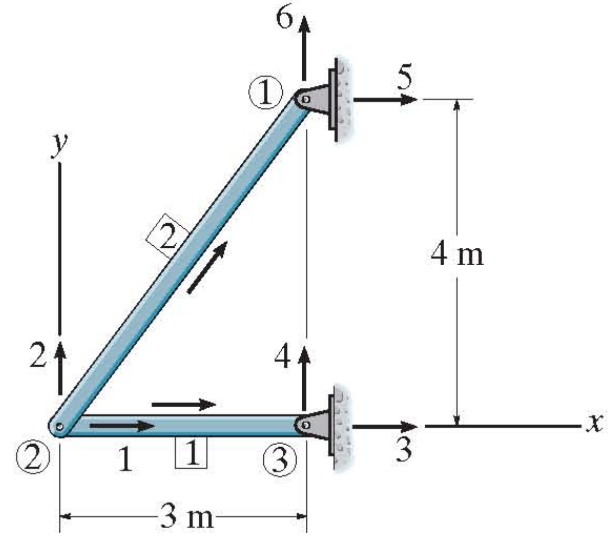
Member 2

- Since 2 is the near end & 1 is the far end, we have:

$$\lambda_x = \frac{3-0}{5} = 0.6; \quad \lambda_y = \frac{4-0}{5} = 0.8$$

- dividing each element by L = 5m, we have:

$$\mathbf{k}_2 = AE \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \begin{bmatrix} 0.072 & 0.096 & -0.072 & -0.096 \\ 0.096 & 0.128 & -0.096 & -0.128 \\ -0.072 & -0.096 & 0.072 & 0.096 \\ -0.096 & -0.128 & 0.096 & 0.128 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \end{array}$$



Example 1

This matrix has an order of 6x6 since there are 6 designated degrees of freedom for the truss.

Corresponding elements of the above 2 matrices are added algebraically to form the structure stiffness matrix.

$$\mathbf{K} = AE \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 0.333 & 0 & -0.333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.333 & 0 & 0.333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} + AE \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 0.072 & 0.096 & 0 & 0 & -0.072 & -0.096 \\ 0.096 & 0.128 & 0 & 0 & -0.096 & -0.128 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.072 & -0.096 & 0 & 0 & 0.072 & 0.096 \\ -0.096 & -0.128 & 0 & 0 & 0.096 & 0.128 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$$\mathbf{K} = AE \left[\begin{array}{cccccc} 0.405 & 0.096 & -0.333 & 0 & -0.072 & -0.096 \\ 0.096 & 0.128 & 0 & 0 & -0.096 & -0.128 \\ -0.333 & 0 & 0.333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.072 & -0.096 & 0 & 0 & 0.072 & 0.096 \\ -0.096 & -0.128 & 0 & 0 & 0.096 & 0.128 \end{array}\right]$$