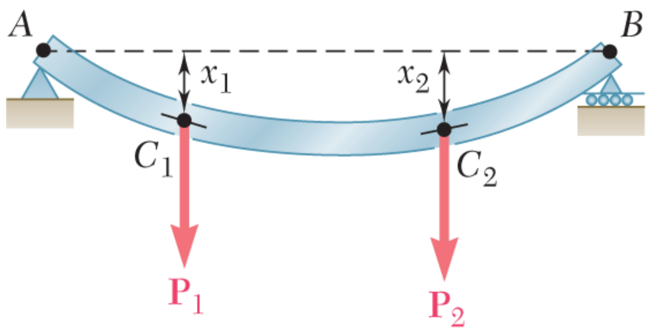
Lecture 8

Energy Method (2)

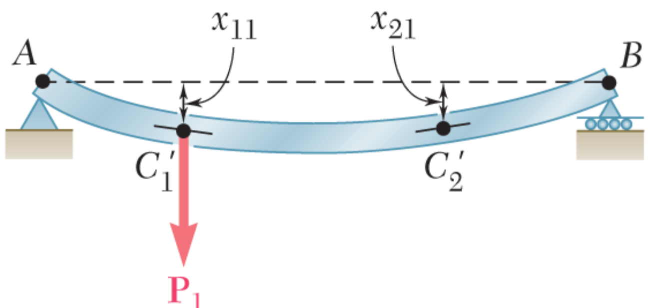
Mohamad Fathi GHANAMEH

Work and Energy Under Several Loads



- Deflections of an elastic beam subjected to two concentrated loads,

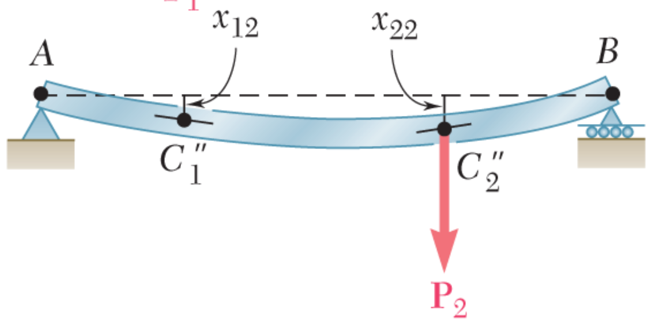
only $\mathbf{P}_1$ is applied to the beam



$$x_{11} = \alpha_{11} P_1 \qquad x_{21} = \alpha_{21} P_1$$

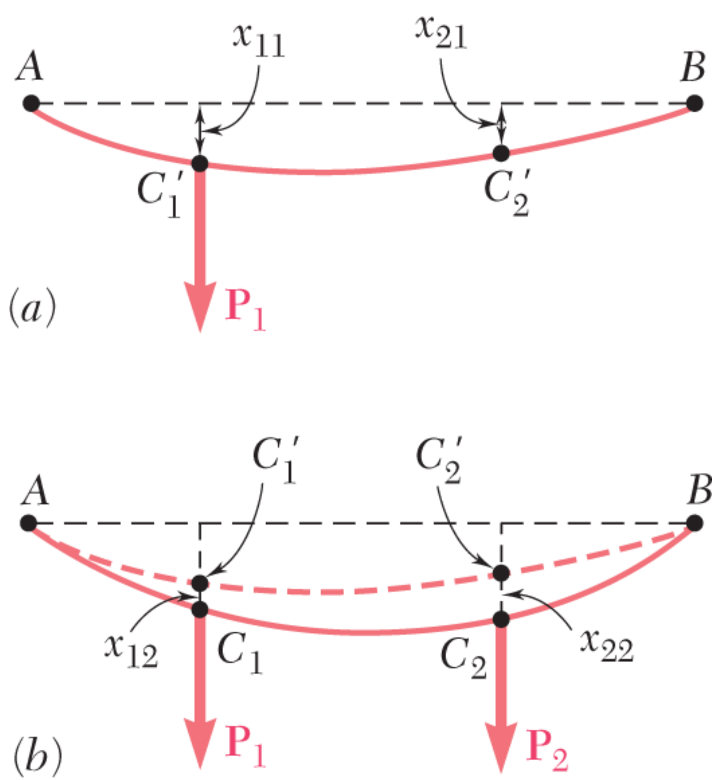
only $\mathbf{P}_2$ is applied to the beam

$$x_{22} = \alpha_{22} P_2 \qquad x_{12} = \alpha_{12} P_2$$



α *influence coefficients*

Work and Energy Under Several Loads



Applying the principle of superposition

$$x_1 = x_{11} + x_{12} = \alpha_{11} P_1 + \alpha_{12} P_2$$

$$x_2 = x_{21} + x_{22} = \alpha_{21} P_1 + \alpha_{22} P_2$$

assume that $\mathbf{P}_1$ is first applied slowly at C_1

$$\tfrac{1}{2} . P_1 . x_{11} = \tfrac{1}{2} \alpha_{11} P_1^2$$

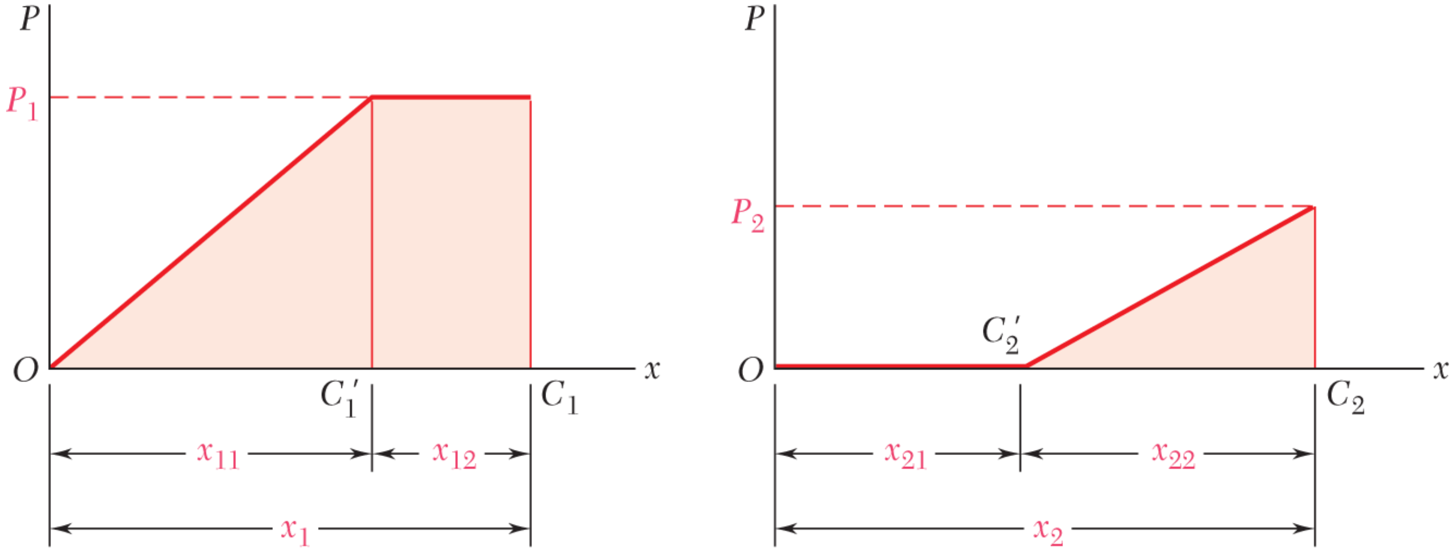
Now slowly apply $\mathbf{P}_2$ at C_2

$$\tfrac{1}{2} . P_2 . x_{22} = \tfrac{1}{2} \alpha_{22} P_2^2$$

Work and Energy Under Several Loads

as $\mathbf{P}_2$ is slowly applied at C_2, the point of application of $\mathbf{P}_1$ moves through x_{12} from C'_1 to C_1, and the load $\mathbf{P}_1$ does work. Since $\mathbf{P}_1$ is *fully applied* during this displacement, its work is equal to $P_1 x_{12}$

$$P_1 . x_{12} = \alpha_{12} P_1 . P_2$$



(*a*) Load-displacement diagram for C_1

(*b*) Load-displacement diagram for C_2

Work and Energy Under Several Loads

- Compute the strain energy in the beam by evaluating the work done by slowly applying P_1 followed by P_2,

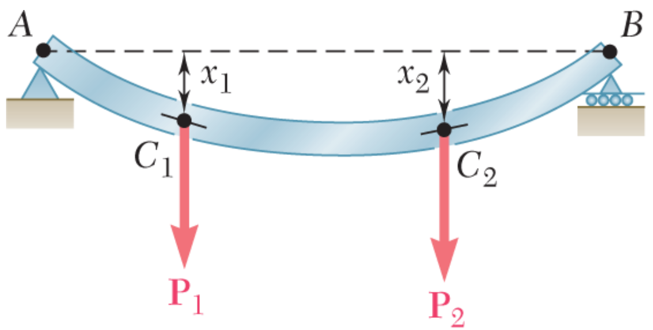
$$U = \frac{1}{2}\left(\alpha_{11}P_1^2 + 2\alpha_{12}P_1P_2 + \alpha_{22}P_2^2\right)$$

- Reversing the application sequence yields

$$U = \frac{1}{2}\left(\alpha_{22}P_2^2 + 2\alpha_{21}P_2P_1 + \alpha_{11}P_1^2\right)$$

- Strain energy expressions must be equivalent. It follows that $\alpha_{12}=\alpha_{21}$ (*Maxwell's reciprocal theorem*).

Castigliano's Theorem



- Strain energy for any elastic structure subjected to two concentrated loads,

$$U = \frac{1}{2}\left(\alpha_{11}P_1^2 + 2\alpha_{12}P_1P_2 + \alpha_{22}P_2^2\right)$$

- Differentiating with respect to the loads,

$$\frac{\partial U}{\partial P_1} = \alpha_{11}P_1 + \alpha_{12}P_2 = x_1$$

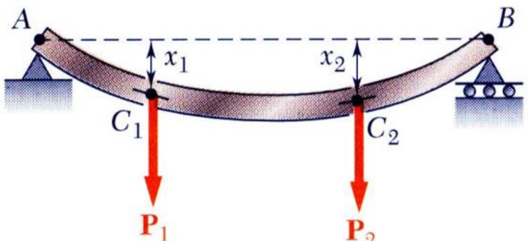
$$\frac{\partial U}{\partial P_2} = \alpha_{12}P_1 + \alpha_{22}P_2 = x_2$$

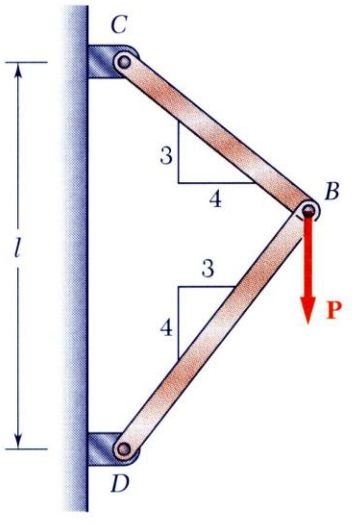
- *Castigliano's theorem*: For an elastic structure subjected to n loads, the deflection x_j of the point of application of P_j can be expressed as

$$x_j = \frac{\partial U}{\partial P_j} \quad \text{and} \quad \theta_j = \frac{\partial U}{\partial M_j} \qquad \phi_j = \frac{\partial U}{\partial T_j}$$

Deflections by Castigliano's Theorem

- Application of Castigliano's theorem is simplified if the differentiation with respect to the load P_j is performed before the integration or summation to obtain the strain energy U.





- In the case of a beam,

$$U = \int_0^L \frac{M^2}{2EI} dx \qquad x_j = \frac{\partial U}{\partial P_j} = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial P_j} dx$$

- For a truss,

$$U = \sum_{i=1}^{n} \frac{F_i^2 L_i}{2 A_i E} \qquad x_j = \frac{\partial U}{\partial P_j} = \sum_{i=1}^{n} \frac{F_i L_i}{A_i E} \frac{\partial F_i}{\partial P_j}$$

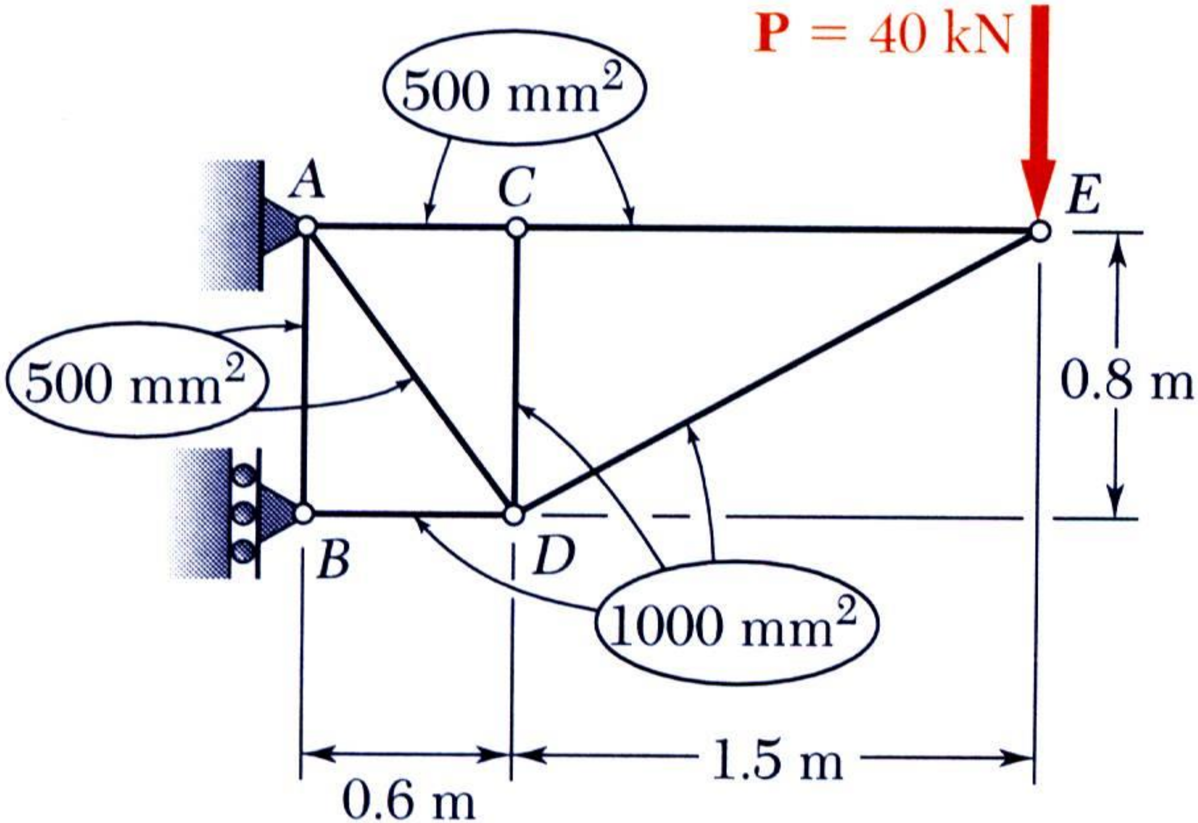
Statically Indeterminate Structures

The reactions at the supports of a statically indeterminate elastic structure can be determined by Castigliano's theorem.

In the case of a structure indeterminate to the first degree, for example, we designate one of the reactions as redundant and eliminate or modify accordingly the corresponding support. The redundant reaction is then treated as an unknown load that, together with the other loads, must produce deformations that are compatible with the original supports. We first calculate the strain energy U of the structure due to the combined action of the given loads and the redundant reaction. Observing that the partial derivative of U with respect to the redundant reaction represents the deflection (or slope) at the support that has been eliminated or modified, we then set this derivative equal to zero and solve the equation obtained for the redundant reaction.† The remaining reactions can be obtained from the equations of statics.

Sample Problem

Members of the truss shown consist of sections of aluminum pipe with the cross-sectional areas indicated. Using E = 73 GPa, determine the vertical deflection of the joint C caused by the load P.



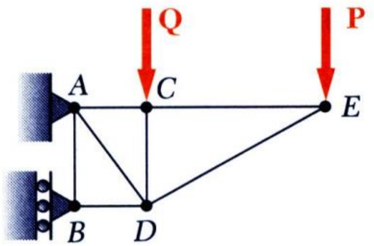
Sample Problem

Application procedure

SOLUTION:

- For application of Castigliano's theorem, introduce a dummy vertical load Q at *the point*. Find the reactions at *support* due to the dummy load from a free-body diagram of the entire truss.
- Apply the method of joints to determine the axial force in each member due to Q.
- Combine with the results of Sample Problem to evaluate the derivative with respect to Q of the strain energy of the truss due to the loads P and Q.
- Setting $Q = 0$, evaluate the derivative which is equivalent to the desired displacement at C.

Sample Problem

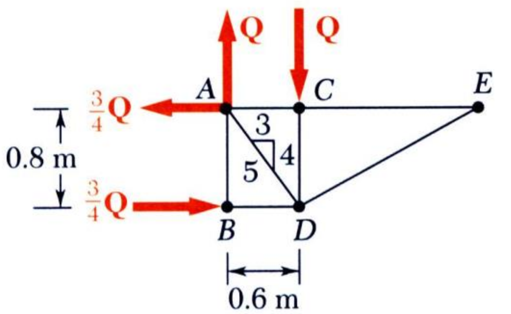


SOLUTION:

- Find the reactions at *A* and *B* due to a dummy load *Q* at *C* from a free-body diagram of the entire truss.

$$A_x = -\tfrac{3}{4}Q \qquad A_y = Q \qquad B = \tfrac{3}{4}Q$$

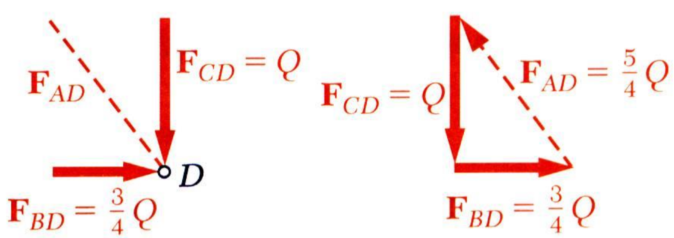
- Apply the method of joints to determine the axial force in each member due to *Q*.



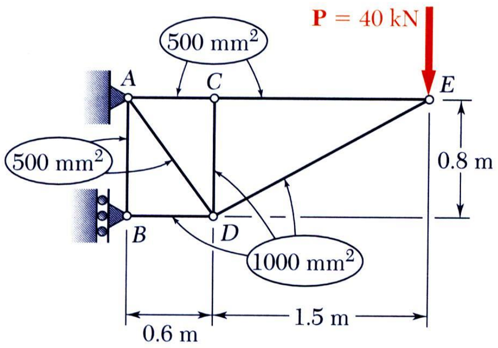
$$F_{CE} = F_{DE} = 0$$

$$F_{AC} = 0;\ F_{CD} = -Q$$

$$F_{AB} = 0;\ F_{BD} = -\tfrac{3}{4}Q$$



Sample Problem



Member	F_i	$\partial F_i/\partial Q$	L_i, m	A_i, m²	$\left(\frac{F_i L_i}{A_i}\right)\frac{\partial F_i}{\partial Q}$
AB	0	0	0.8	500×10^{-6}	0
AC	$+15P/8$	0	0.6	500×10^{-6}	0
AD	$+5P/4 + 5Q/4$	$\frac{5}{4}$	1.0	500×10^{-6}	$+3125P + 3125Q$
BD	$-21P/8 - 3Q/4$	$-\frac{3}{4}$	0.6	1000×10^{-6}	$+1181P + 338Q$
CD	$-Q$	-1	0.8	1000×10^{-6}	$+800Q$
CE	$+15P/8$	0	1.5	500×10^{-6}	0
DE	$-17P/8$	0	1.7	1000×10^{-6}	0

- Combine with the results of Sample Problem 11.4 to evaluate the derivative with respect to *Q* of the strain energy of the truss due to the loads *P* and *Q*.

$$y_C = \sum\left(\frac{F_i L_i}{A_i E}\right)\frac{\partial F_i}{\partial Q} = \frac{1}{E}(4306P + 4263Q)$$

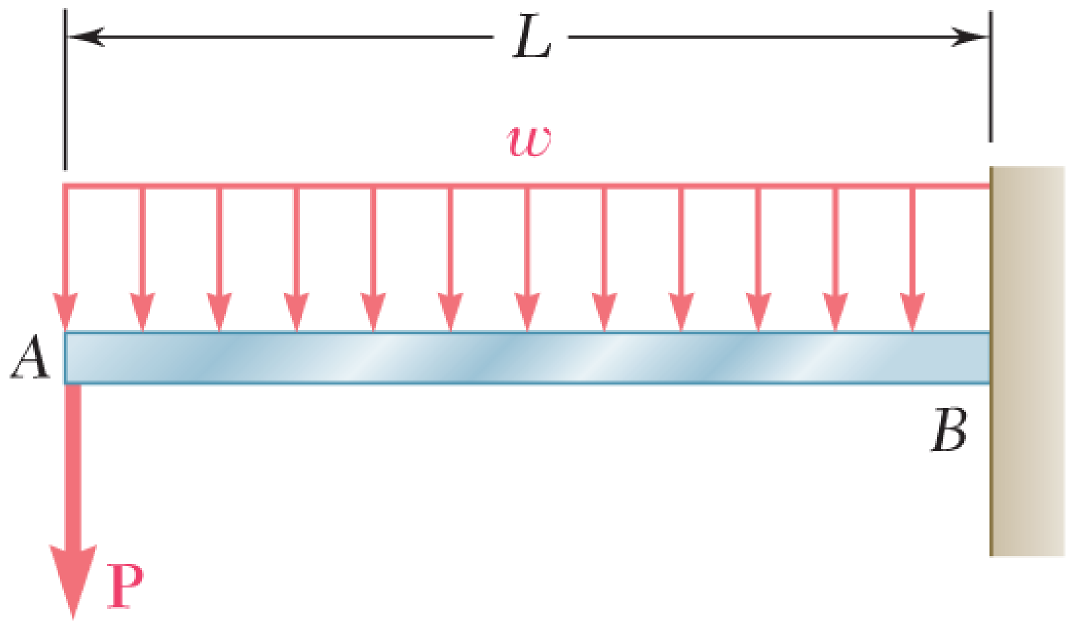
- Setting $Q = 0$, evaluate the derivative which is equivalent to the desired displacement at *C*.

$$y_C = \frac{4306\left(40\times 10^3\,N\right)}{73\times 10^9\,\text{Pa}}$$

$$\boxed{y_C = 2.36\,\text{mm}\downarrow}$$

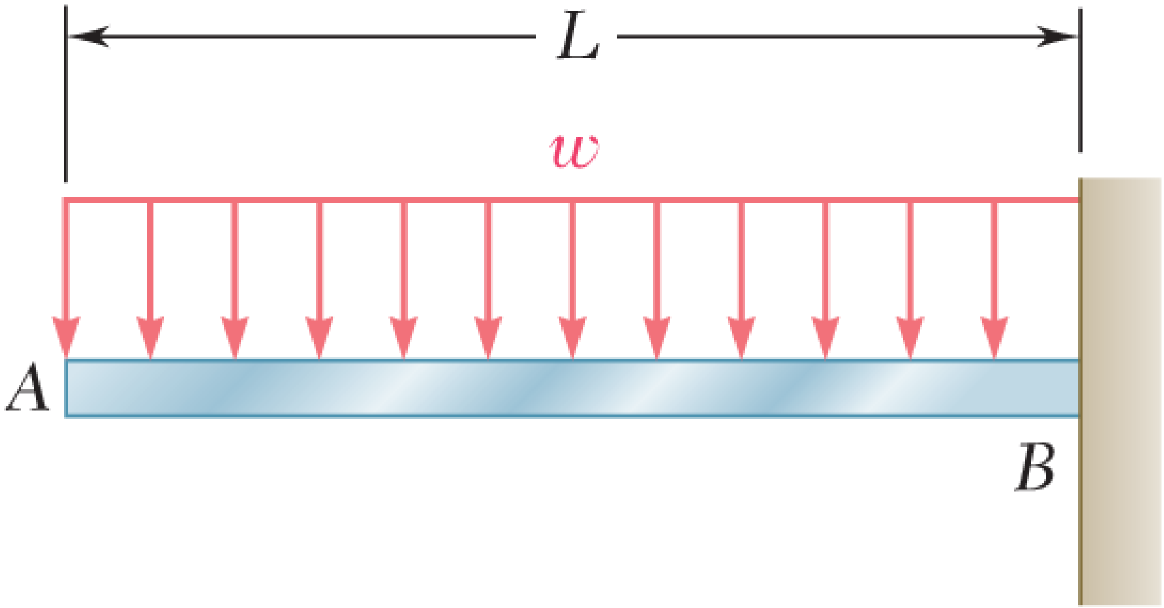
Example 1

The cantilever beam AB supports a uniformly distributed load w and a concentrated load P as shown. Knowing that $L = 2\ m$, $w = 4\ kN/m$, $P = 6\ kN$, and $E.I = 5\ MN.m^2$, determine the deflection at A.



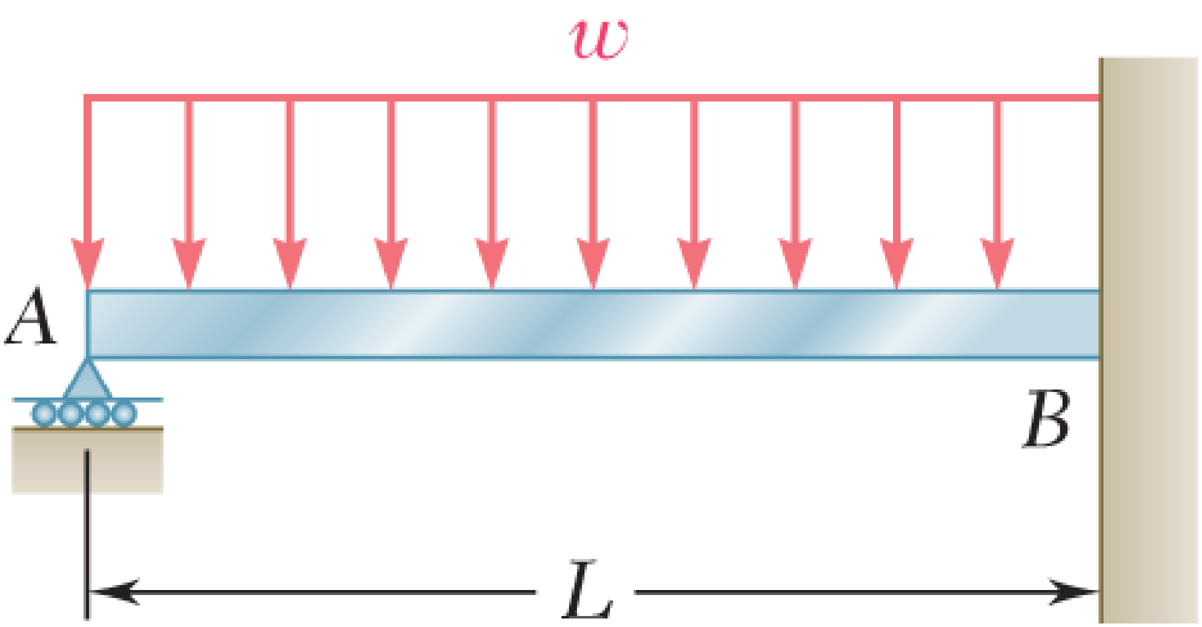
Example 2

The cantilever beam AB supports a uniformly distributed load w as shown. Determine the deflection and slop at A.



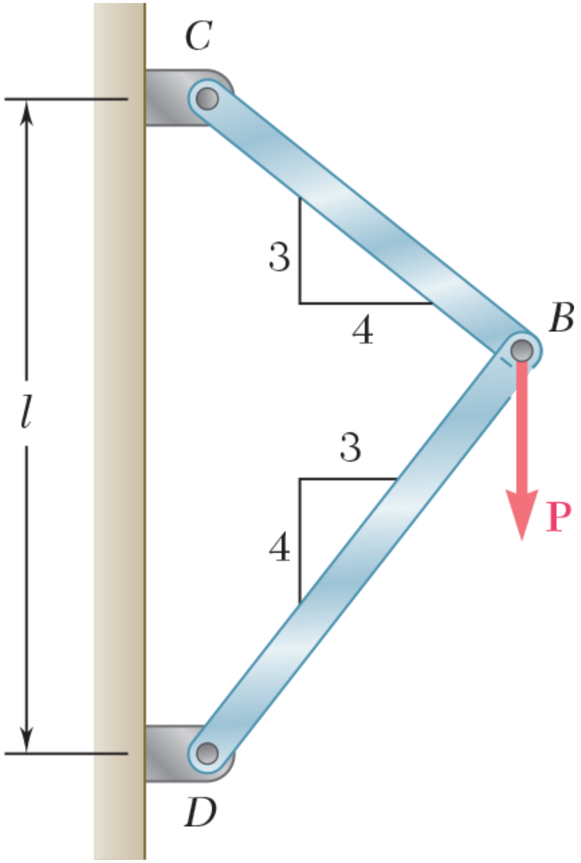
Example 3

The cantilever beam AB supports a uniformly distributed load w as shown. Determine the deflection and slop at A.



Example 4

A load P is supported at B by two rods of the same material and of the same cross-sectional area A . Determine the horizontal and vertical deflection of point B.



Example 5

A load P is supported at B by three rods of the same material and the same cross-sectional area A. Determine the force in each rod.