Structural Analysis

Lecture 7

Energy Method

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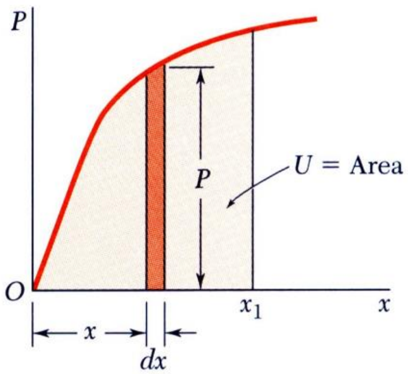
Strain Energy

- A uniform rod is subjected to a slowly increasing load



- The *elementary work* done by the load P as the rod elongates by a small *dx* is

$$dU = P\,dx = \text{elementary work}$$

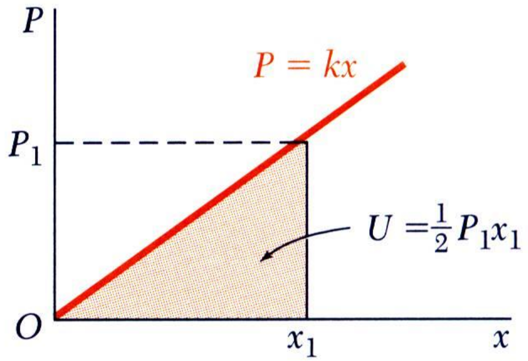
- The *total work* done by the load for a deformation *y*,

$$U = \int_0^x P\,dx = \text{total work} = \text{strain energy}$$

which is equal to the area under the load–deflection curve.

- which results in an increase of *strain energy* in the rod.

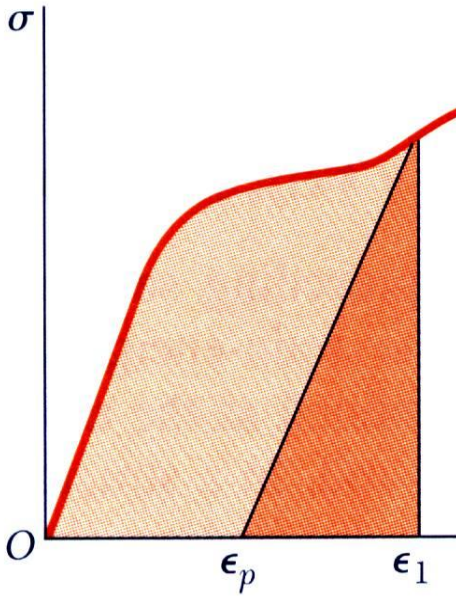
Strain Energy



- In the case of a linear elastic deformation,

$$U = \int_0^{x_1} kx\,dx = \frac{1}{2}kx_1^2 = \frac{1}{2}P_1x_1$$

Strain Energy Density



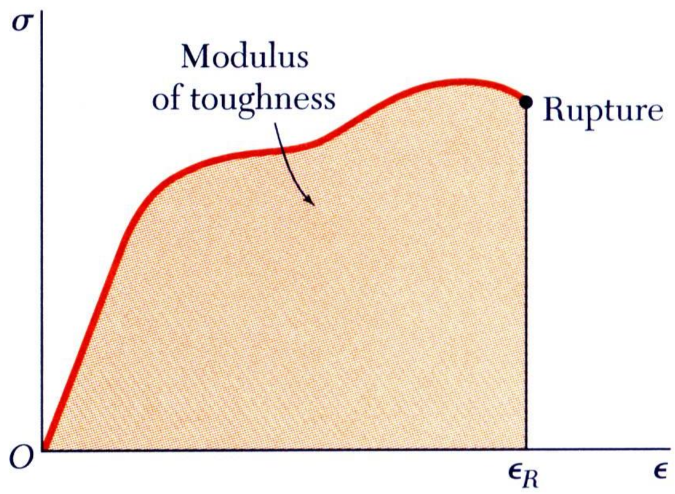
- To eliminate the effects of size, evaluate the strain-energy per unit volume,

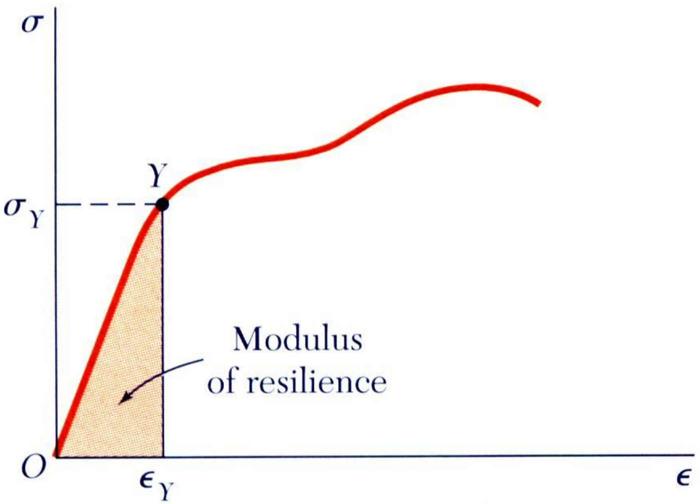
$$\frac{U}{V} = \int_0^{x_1} \frac{P}{A} \frac{dx}{L}$$

$$u = \int_0^{\varepsilon_1} \sigma_x \, d\varepsilon = strain\, energy\, density$$

- The total strain energy density resulting from the deformation is equal to the area under the curve to ε_1.
- As the material is unloaded, the stress returns to zero but there is a permanent deformation. Only the strain energy represented by the triangular area is recovered.
- Remainder of the energy spent in deforming the material is dissipated as heat.

Strain Energy Density





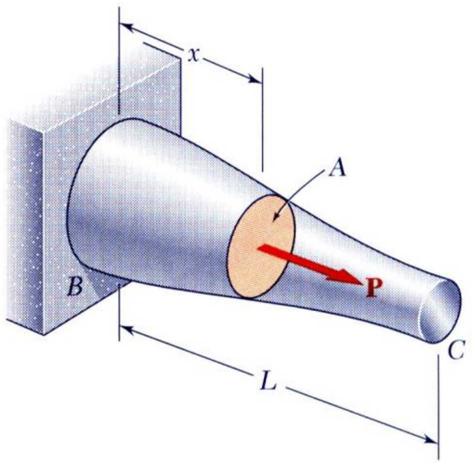
- The strain energy density resulting from setting $\varepsilon_1 = \varepsilon_R$ is the *modulus of toughness*.
- The energy per unit volume required to cause the material to rupture is related to its ductility as well as its ultimate strength.
- If the stress remains within the proportional limit,

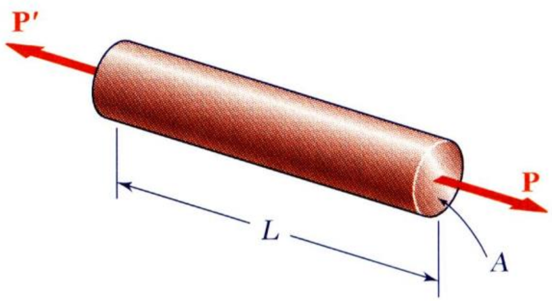
$$u = \int_0^{\varepsilon_1} E\varepsilon_1 \, d\varepsilon_x = \frac{E\varepsilon_1^2}{2} = \frac{\sigma_1^2}{2E}$$

- The strain energy density resulting from setting $\sigma_1 = \sigma_Y$ is the *modulus of resilience*.

$$u_Y = \frac{\sigma_Y^2}{2E} = modulus\ of\ resilience$$

Elastic Strain Energy for Normal Stresses





- In an element with a non-uniform stress distribution,

$$u = \lim_{\Delta V \to 0} \frac{\Delta U}{\Delta V} = \frac{dU}{dV} \qquad U = \int u \, dV = \text{total strain energy}$$

- For values of $u < u_Y$, i.e., below the proportional limit,

$$U = \int \frac{\sigma_x^2}{2E} dV = \textit{elastic strain energy}$$

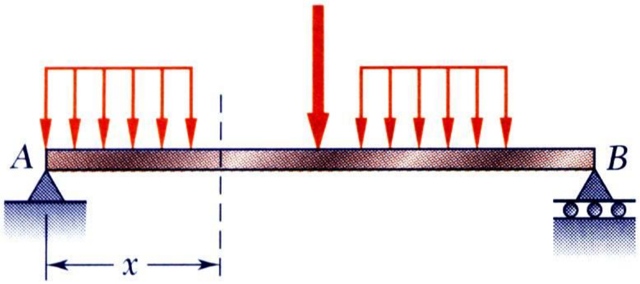
- Under axial loading, $\sigma_x = P/A \qquad dV = A\,dx$

$$U = \int_0^L \frac{P^2}{2AE} dx$$

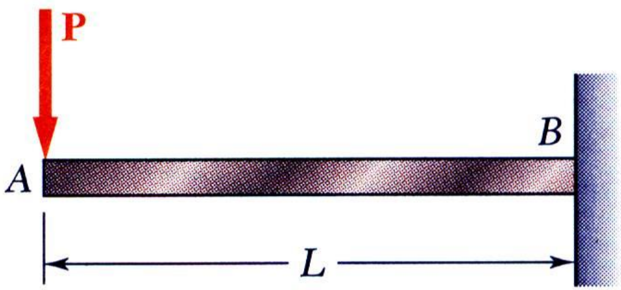
- For a rod of uniform cross-section,

$$U = \frac{P^2 L}{2AE}$$

Elastic Strain Energy for Normal Stresses



$$\sigma_x = \frac{M\,y}{I}$$



- For a beam subjected to a bending load,

$$U = \int \frac{\sigma_x^2}{2E} dV = \int \frac{M^2 y^2}{2EI^2} dV$$

- Setting $dV = dA\,dx$,

$$U = \int_0^L \int_A \frac{M^2 y^2}{2EI^2} dA\,dx = \int_0^L \frac{M^2}{2EI^2}\left(\int_A y^2 dA\right) dx$$

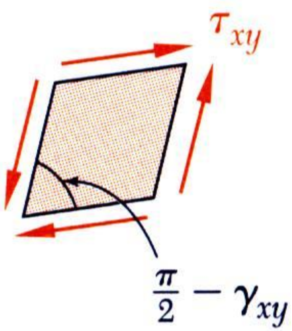
$$= \int_0^L \frac{M^2}{2EI} dx$$

- For an end-loaded cantilever beam,

$$M = -Px$$

$$U = \int_0^L \frac{P^2 x^2}{2EI} dx = \frac{P^2 L^3}{6EI}$$

Strain Energy For Shearing Stresses

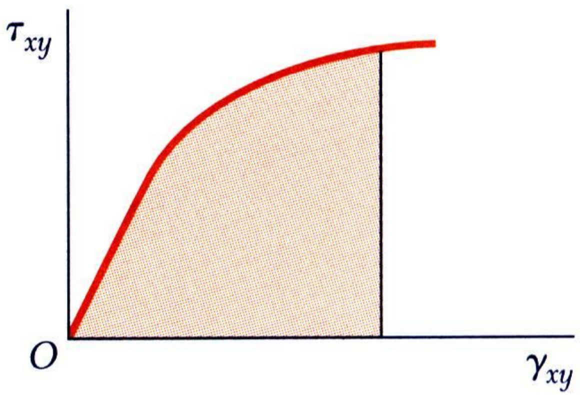


- For a material subjected to plane shearing stresses,

$$u = \int_0^{\gamma_{xy}} \tau_{xy}\, d\gamma_{xy}$$

- For values of τ_{xy} within the proportional limit,

$$u = \tfrac{1}{2} G \gamma_{xy}^2 = \tfrac{1}{2} \tau_{xy} \gamma_{xy} = \frac{\tau_{xy}^2}{2G}$$

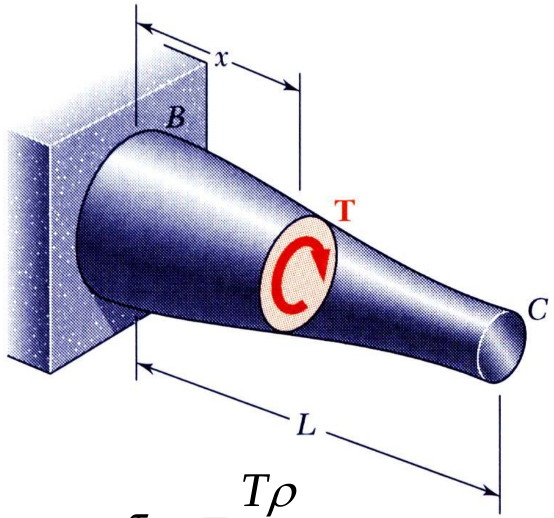


- The total strain energy is found from

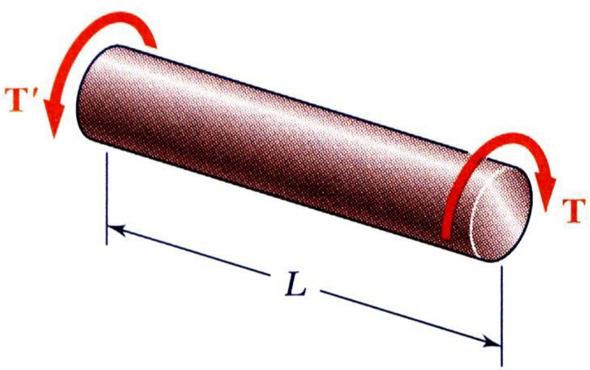
$$U = \int u\, dV$$

$$= \int \frac{\tau_{xy}^2}{2G}\, dV$$

Strain Energy For Shearing Stresses



$$\tau_{xy} = \frac{T\rho}{J}$$



- For a shaft subjected to a torsional load,

$$U = \int \frac{\tau_{xy}^2}{2G} dV = \int \frac{T^2 \rho^2}{2GJ^2} dV$$

- Setting $dV = dA\, dx$,

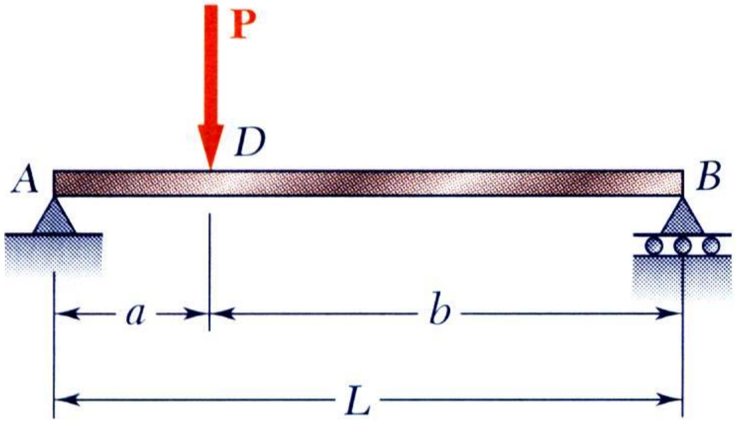
$$U = \int_0^L \int_A \frac{T^2 \rho^2}{2GJ^2} dA\, dx = \int_0^L \frac{T^2}{2GJ^2} \left(\int_A \rho^2 dA \right) dx$$

$$= \int_0^L \frac{T^2}{2GJ} dx$$

- In the case of a uniform shaft,

$$U = \frac{T^2 L}{2GJ}$$

Sample Problem 1

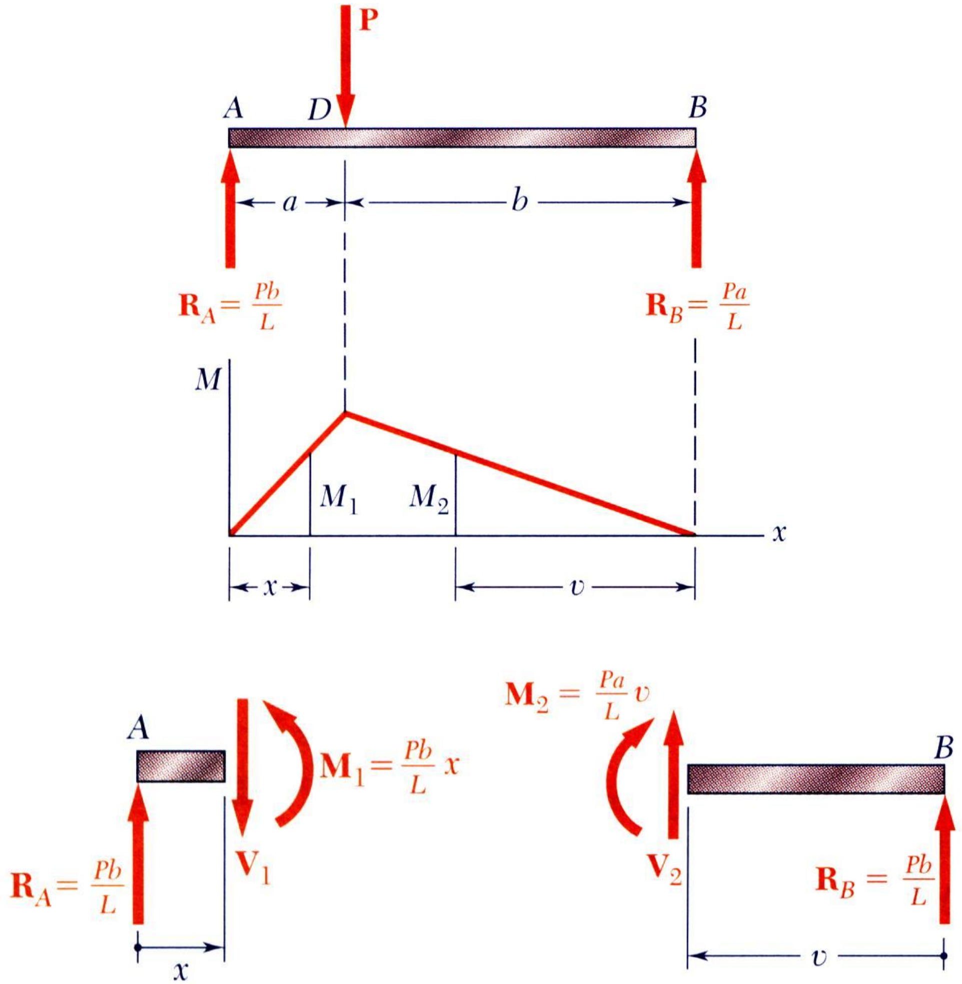


Taking into account only the normal stresses due to bending, determine the strain energy of the beam for the loading shown.

SOLUTION:

- Determine the reactions at *A* and *B* from a free-body diagram of the complete beam.
- Develop a diagram of the bending moment distribution.
- Integrate over the volume of the beam to find the strain energy.
- Apply the particular given conditions to evaluate the strain energy.

Sample Problem 1



SOLUTION:

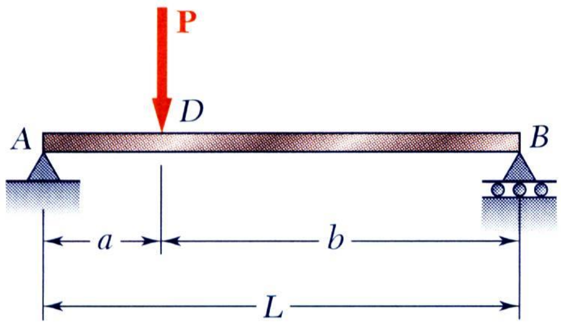
- Determine the reactions at *A* and *B* from a free-body diagram of the complete beam.

$$R_A = \frac{Pb}{L} \qquad R_B = \frac{Pa}{L}$$

- Develop a diagram of the bending moment distribution.

$$M_1 = \frac{Pb}{L} x \qquad M_2 = \frac{Pa}{L} v$$

Sample Problem 1



Over the portion AD,

$$M_1 = \frac{Pb}{L}x$$

Over the portion BD,

$$M_2 = \frac{Pa}{L}v$$

- Integrate over the volume of the beam to find the strain energy.

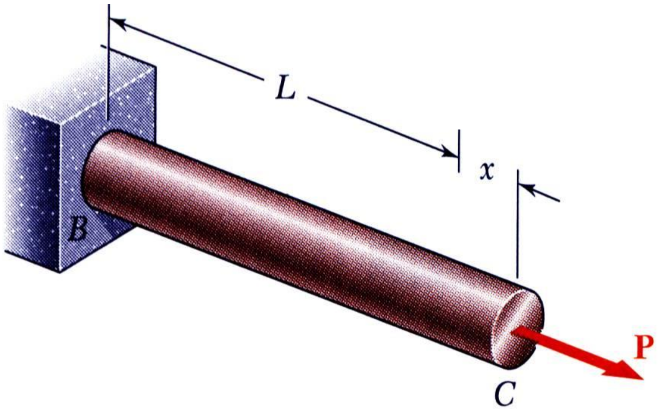
$$U = \int_0^a \frac{M_1^2}{2EI}dx + \int_0^b \frac{M_2^2}{2EI}dv$$

$$= \frac{1}{2EI}\int_0^a \left(\frac{Pb}{L}x\right)^2 dx + \frac{1}{2EI}\int_0^b \left(\frac{Pa}{L}x\right)^2 dx$$

$$= \frac{1}{2EI}\frac{P^2}{L^2}\left(\frac{b^2a^3}{3} + \frac{a^2b^3}{3}\right) = \frac{P^2a^2b^2}{6EIL^2}(a+b)$$

$$U = \frac{P^2a^2b^2}{6EIL}$$

Work and Energy Under a Single Load



- Previously, we found the strain energy by integrating the energy density over the volume. For a uniform rod,

$$U = \int u\,dV = \int \frac{\sigma^2}{2E}\,dV$$

$$= \int_0^L \frac{(P_1/A)^2}{2E}\,A dx = \frac{P_1^2 L}{2AE}$$

- Strain energy may also be found from the work of the single load P_1,

$$U = \int_0^{x_1} P\,dx$$

- For an elastic deformation,

$$U = \int_0^{x_1} P\,dx = \int_0^{x_1} kx\,dx = \tfrac{1}{2}k x_1^2 = \tfrac{1}{2}P_1 x_1$$

- Knowing the relationship between force and displacement,

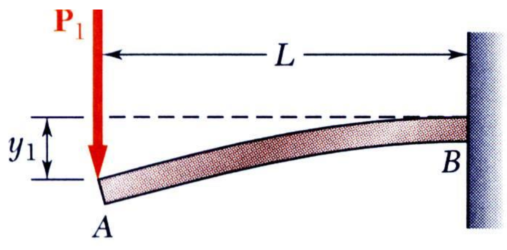
$$x_1 = \frac{P_1 L}{AE}$$

$$U = \tfrac{1}{2}P_1\left(\frac{P_1 L}{AE}\right) = \frac{P_1^2 L}{2AE}$$

Work and Energy Under a Single Load

- Strain energy may be found from the work of other types of single concentrated loads.

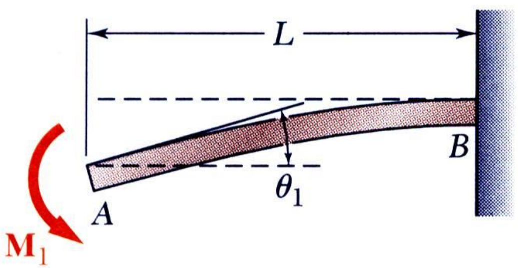
- Transverse load



$$U = \int_0^{y_1} P\,dy = \tfrac{1}{2}P_1 y_1$$

$$= \tfrac{1}{2}P_1\left(\frac{P_1 L^3}{3EI}\right) = \frac{P_1^2 L^3}{6EI}$$

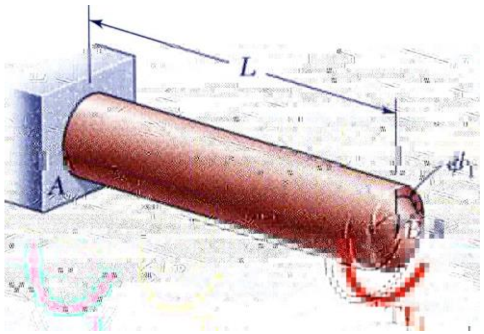
- Bending couple



$$U = \int_0^{\theta_1} M\,d\theta = \tfrac{1}{2}M_1\theta_1$$

$$= \tfrac{1}{2}M_1\left(\frac{M_1 L}{EI}\right) = \frac{M_1^2 L}{2EI}$$

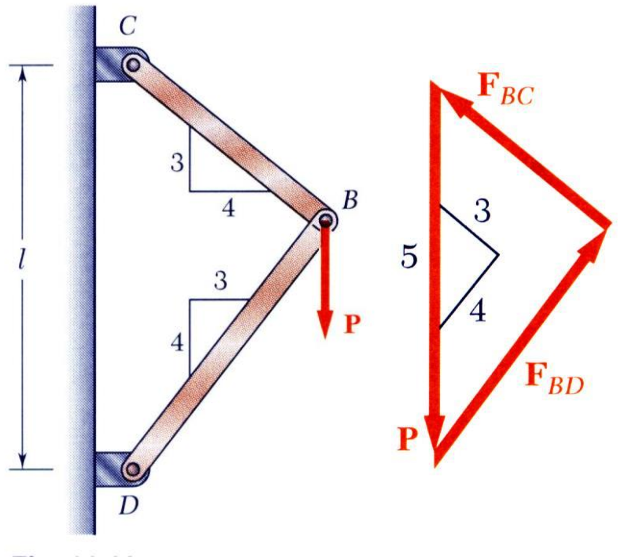
- Torsional couple



$$U = \int_0^{\phi_1} T\,d\phi = \tfrac{1}{2}T_1\phi_1$$

$$= \tfrac{1}{2}T_1\left(\frac{T_1 L}{JG}\right) = \frac{T_1^2 L}{2JG}$$

Sample Problem 2



From the given geometry,

$$L_{BC} = 0.6l \quad L_{BD} = 0.8l$$

From statics,

$$F_{BC} = +0.6P \quad F_{BD} = -0.8P$$

- If the strain energy of a structure due to a single concentrated load is known, then the equality between the work of the load and energy may be used to find the deflection.
  - Strain energy of the structure,

$$U = \frac{F_{BC}^2 L_{BC}}{2AE} + \frac{F_{BD}^2 L_{BD}}{2AE}$$

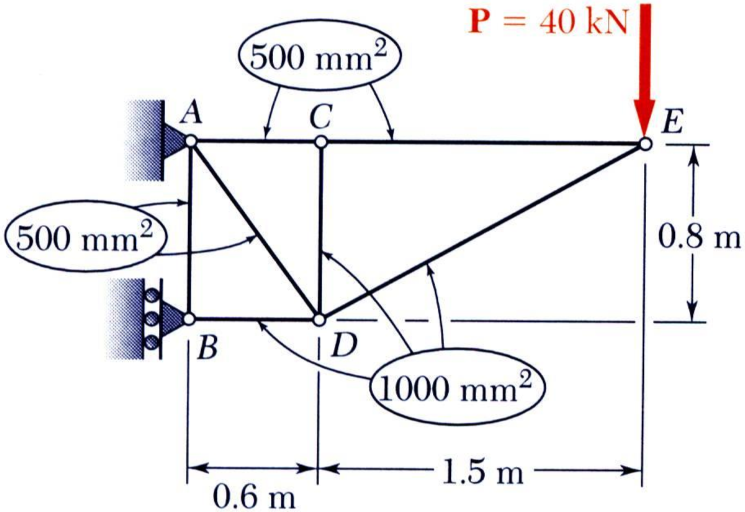
$$= \frac{P^2 l\left[(0.6)^3 + (0.8)^3\right]}{2AE} = 0.364\frac{P^2 l}{AE}$$

  - Equating work and strain energy,

$$U = 0.364\frac{P^2 L}{AE} = \tfrac{1}{2} P y_B$$

$$y_B = 0.728\frac{Pl}{AE}$$

Sample Problem 3

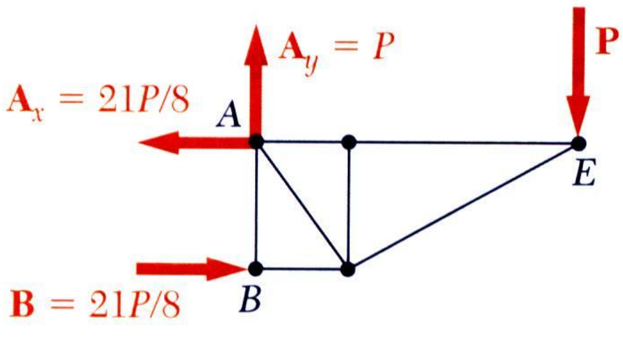


Members of the truss shown consist of sections of aluminum pipe with the cross-sectional areas indicated. Using E = 73 GPa, determine the vertical deflection of the point E caused by the load P.

SOLUTION:

- Find the reactions at A and B from a free-body diagram of the entire truss.
- Apply the method of joints to determine the axial force in each member.
- Evaluate the strain energy of the truss due to the load P.
- Equate the strain energy to the work of P and solve for the displacement.

Sample Problem 3

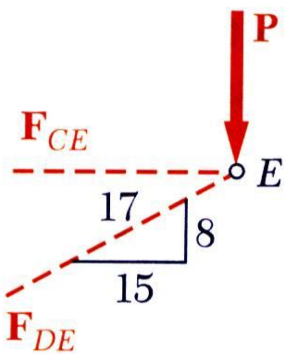


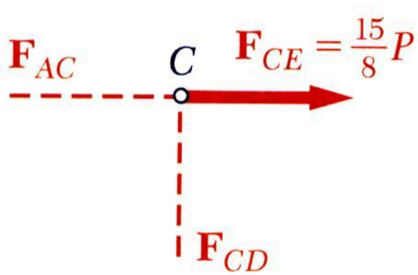
SOLUTION:

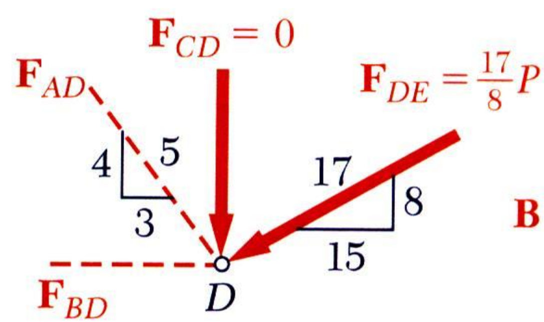
- Find the reactions at A and B from a free-body diagram of the entire truss.

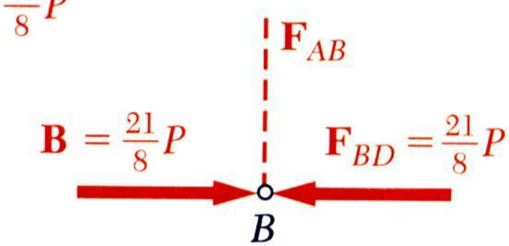
$$A_x = -21P/8 \qquad A_y = P \qquad B = 21P/8$$

- Apply the method of joints to determine the axial force in each member.





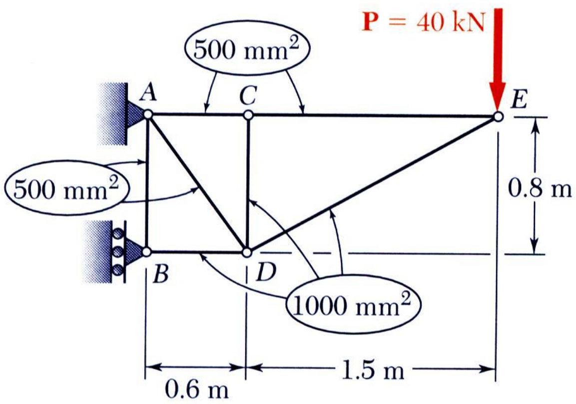




$$F_{DE} = -\frac{17}{8}P \qquad F_{AC} = +\frac{15}{8}P \qquad F_{DE} = \frac{5}{4}P \qquad F_{AB} = 0$$

$$F_{CE} = +\frac{15}{8}P \qquad F_{CD} = 0 \qquad F_{CE} = -\frac{21}{8}P$$

Sample Problem 3



Member	F_i	L_i, m	A_i, m^2	$\frac{F_i^2 L_i}{A_i}$
AB	0	0.8	500×10^{-6}	0
AC	$+15P/8$	0.6	500×10^{-6}	$4\,219P^2$
AD	$+5P/4$	1.0	500×10^{-6}	$3\,125P^2$
BD	$-21P/8$	0.6	1000×10^{-6}	$4\,134P^2$
CD	0	0.8	1000×10^{-6}	0
CE	$+15P/8$	1.5	500×10^{-6}	$10\,547P^2$
DE	$-17P/8$	1.7	1000×10^{-6}	$7\,677P^2$

- Evaluate the strain energy of the truss due to the load *P*.

$$U = \sum \frac{F_i^2 L_i}{2A_i E} = \frac{1}{2E}\sum \frac{F_i^2 L_i}{A_i}$$

$$= \frac{1}{2E}\left(29700P^2\right)$$

- Equate the strain energy to the work by *P* and solve for the displacement.

$$\tfrac{1}{2} P y_E = U$$

$$y_E = \frac{2U}{P} = \frac{2}{P}\left(\frac{29700P^2}{2E}\right)$$

$$y_E = \frac{\left(29.7\times10^3\right)\left(40\times10^3\right)}{73\times10^9}$$

$$y_E = 16.27\text{mm} \downarrow$$