Lecture 6

Virtual Work Method (2)

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Virtual Work for Deformable Body

Deformable Body

→ Structures and structural members deform

→ Different points of a structure will suffer different virtual displacements

→ Virtual work produced by the internal forces is not zero

Virtual Work for Deformable Body

The total virtual work produced by applying a virtual displacement to a deformable body acted on by a system of external forces is therefore given by:

$$W_t = W_e + W_i$$

If the body is in equilibrium under the action of the external force system, then every particle in the body is also in equilibrium.
Therefore, the virtual work is zero for all particles in the body, thus it is zero for the complete body

$$0 = W_e + W_i$$

Only the conditions of equilibrium and the concept of work are employed.
⇒ it's not require the deformable body to be linearly elastic

Virtual Work for Deformable Body

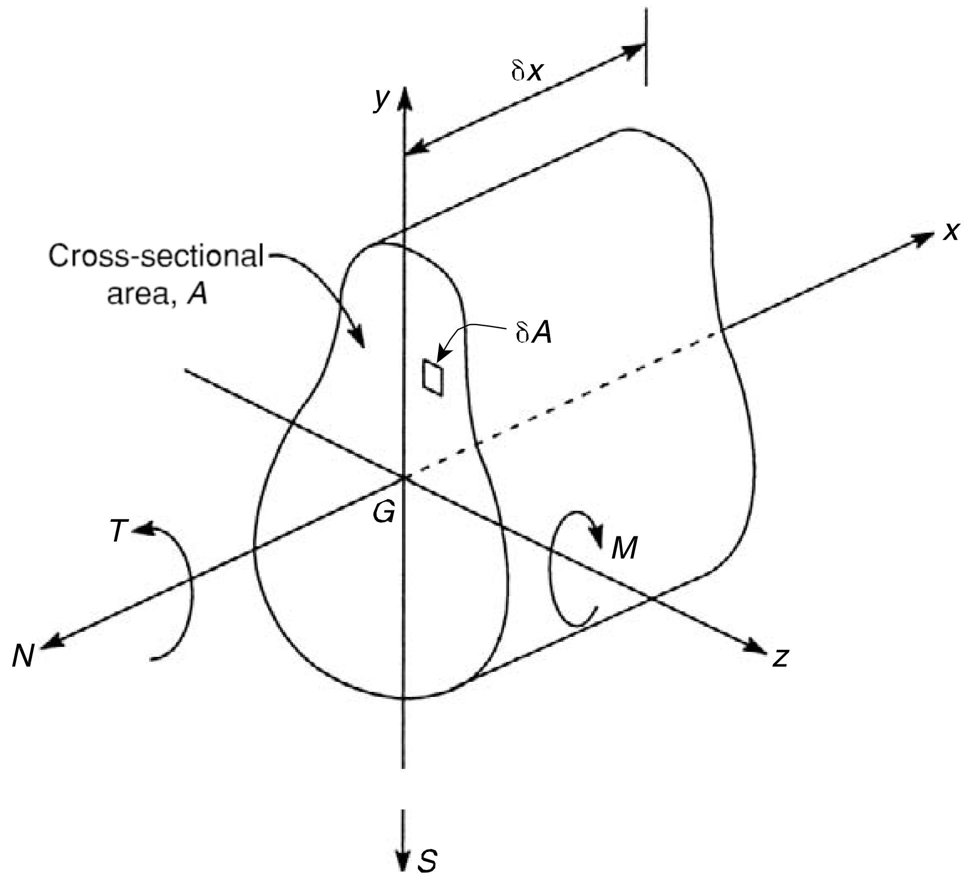
virtual work may be applied to any body or structure that is rigid, elastic, or plastic. The principle does require that displacements, whether real or imaginary, must be small, so that we may assume that external and internal forces are unchanged in magnitude and direction during the displacements. In addition, the virtual displacements must be compatible with the geometry of the structure and the constraints that are applied, such as those at a support.

Work Done by Internal Force Systems

The calculation of the work done by an external force is straightforward in that it is the product of the force and the displacement of its point of application in its own line of action, whereas the calculation of the work done by an internal force system during a displacement is much more complicated. Generally, no matter how complex a loading system is, it may be simplified to a combination of up to four load types: axial load, shear force, bending moment, and torsion; these in turn produce corresponding internal force systems.

Work Done by Internal Force Systems

Consider the elemental length, δ*x*, of a structural member, and suppose that it is subjected to a positive internal force system comprising a normal force *N*; a shear force, *S*; a bending moment, *M*; and a torque, *T*, produced by some external loading system acting on the structure of which the member is part.



Axial Force

The direct stress σ_x at any point in the cross section of the member is given by $\sigma_x = \frac{N_A}{A}$. Therefore, the normal force on the element δA at the point (z, y) is:

$$\delta N_A = \sigma_x . \delta A = \frac{N_A}{A} \delta A$$

Suppose now that the structure is given an arbitrary virtual displacement Δ_x which produces a virtual axial strain ε_x in the element. The internal virtual work $\delta W_{i,N}$ done by the axial force on the elemental length of the member is given by

$$\delta W_{i,N} = \int_A dN_A . \Delta_x = \int_A \frac{N_A}{A} dA . \varepsilon_v dx$$

since $\int_A \delta A = A$

$$\delta W_{i,N} = N_A \varepsilon_v \delta x$$

Axial Force

For a member of length L

$$W_{i,N} = \int_L dW_{i,N} = \int_L N_A \varepsilon_v dx$$

For a structure comprising a number of members

$$W_{i,N} = \sum \int_L N_A . \varepsilon_x . dx$$

Nonelastic as well as elastic materials

For linearly elastic material

$$\varepsilon_v = \frac{\sigma_v}{E} = \frac{N_v}{A.E}$$

$$W_{i,N} = \sum \int_L \frac{N_A . N_v}{A.E} dx$$

Shear Force

The shear stress τ at any point in the cross section of the member is given by $\tau = \frac{S_A}{A}$.
Therefore, the shear force on the element δA at the point (z, y) is:

$$\delta S_A = \tau \,.\delta A = \beta \frac{S_A}{A} \delta A$$

Suppose now that the structure is given an arbitrary virtual displacement which produces a virtual shear strain γ_{yz} in the element. This shear strain represents the angular rotation in a vertical plane of the element $\delta A \times \delta x$ relative to the longitudinal centroidal axis of the member. The vertical displacement at the section being considered is, therefore, $\gamma_{yz} \times \delta x$. The internal virtual work $\delta W_{i,S}$ done by the shear force on the elemental length of the member is given by

$$\delta W_{i,S} = \int_A \delta S_A .\Delta_x = \int_A \beta \frac{S_A}{A} \delta A .\gamma_v .dx$$

Shear Force

since $\int_A \delta A = A$

For a member of length L

$$\delta W_{i,S} = \beta.\int_L S_A.\gamma_v.dx$$

For a structure comprising a number of members

$$W_{i,S} = \sum \beta.\int_L S_A.\gamma_v.dx$$ Nonelastic as well as elastic materials

For linearly elastic material

$$\gamma_v = \frac{\tau_v}{G} = \frac{S_v}{A.G}$$

$$W_{i,S} = \sum \beta.\int_L \frac{S_A.S_v}{A.G}.dx$$

Bending Moment

The bending moment, M, acting on the member section produces a distribution of direct σ,

The Normal force $\sigma_x . A$

structure is given a small arbitrary virtual displacement which produces a virtual direct strain ε_x, in the element $\delta A \times \delta x$

$$\delta W_{i,\mathrm{M}} = \int_A \sigma . dA . \varepsilon_v dx$$

The virtual strain given by

$$\varepsilon_v = \frac{y}{R_A}$$

$$\delta W_{i,\mathrm{M}} = \int_A \sigma . dA . \frac{y}{R_A} . dx$$

since $\int_A \delta A = A$ moment of the normal force about the *z* axis $M = \sigma . A . y$

Bending Moment

$$\delta W_{i,\mathrm{M}} = \frac{M}{R_A}.\delta x$$

for a member of length L

$$W_{i,\mathrm{M}} = \int_L \frac{M}{R_A}.dx$$

For a structure comprising a number of members

$$W_{i,\mathrm{M}} = \sum \int_L \frac{M}{R_A}.dx$$

Nonelastic as well as elastic materials

Bending Moment

linearly elastic system

virtual curvature $1/R_v$ may be expressed in terms of an equivalent virtual bending moment, M_v,

$$\frac{1}{R_A} = \frac{M_v}{EI}$$

$$W_{i,\mathrm{M}} = \sum \int_L \frac{M_A M_v}{EI} .dx$$

Torsion

$$W_{i,T} = \sum \int_L \frac{T_A . T_v}{G . J_p} . dx$$

Hinges

In some cases, it is convenient to impose a virtual rotation, θ_V, at some point in a structural member where, say, the actual bending moment is M_A. The internal virtual work done by M_A is then $M_A.\theta_V$; physically this situation is equivalent to inserting a hinge at the point.

Work Done by Internal Force Systems

internal virtual work for a linear system is

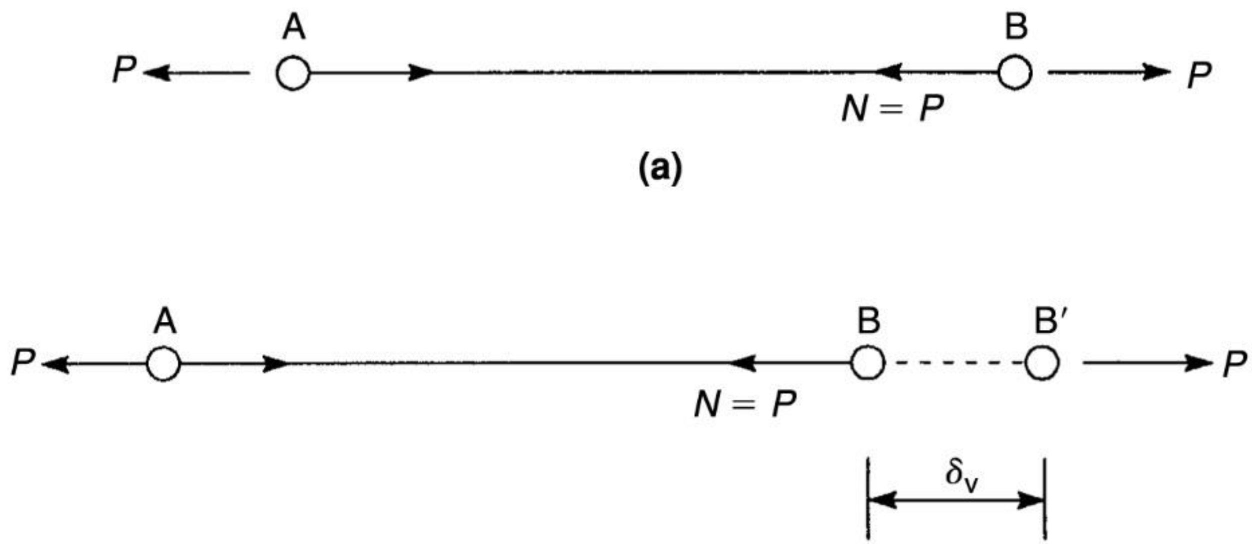
$$W_i = \sum \left[\int_L \frac{N_A . N_v}{A . E} dx + \beta . \int_L \frac{S_A . S_v}{A . G} . \delta x + \int_L \frac{M_A M_v}{EI} . dx + \int_L \frac{T_A . T_v}{G . J_p} . dx + M_A . \theta_v \right]$$

Sign of Internal Virtual Work

member is given a virtual extension

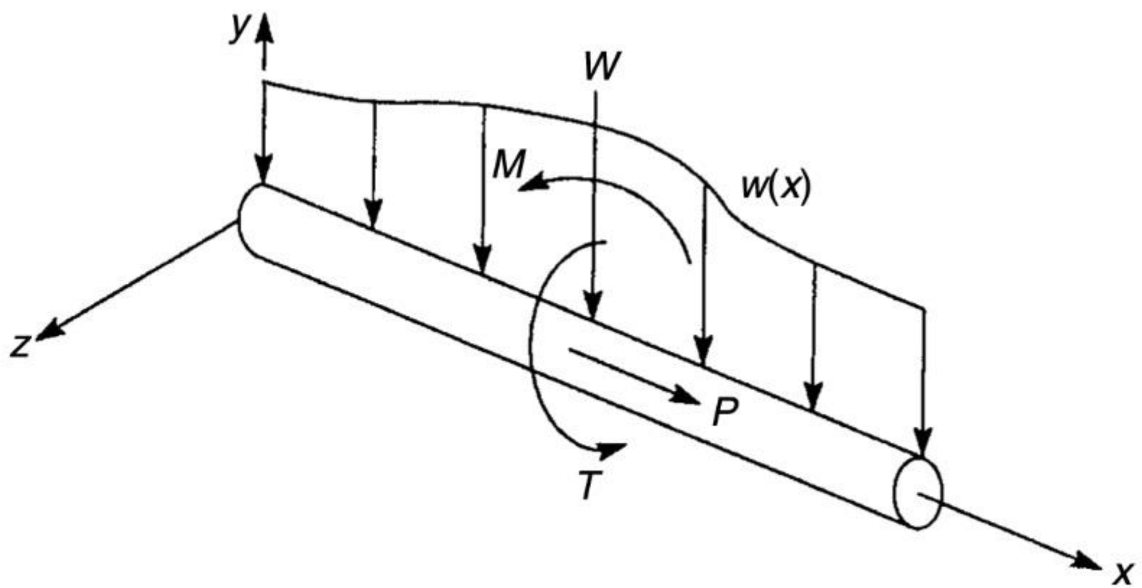
virtual work done by the applied load, *P*, is positive since the displacement, δv, is in the same direction as its line of action. However, the virtual work done by the internal force, *N* (=*P*), is negative, since the displacement of B is in the opposite direction to its line of action

signs of the external and internal virtual work in Eq. (4.8) would have been reversed.



Virtual Work due to External Force Systems

a structural member carries a distributed load, $w(x)$, and at a particular point a concentrated load, W and P; a moment, M; and a torque, T. Suppose that at the point a virtual displacement is imposed that has translational components, $\Delta_{v,y}$ and $\Delta_{v,x}$, parallel to the y and x axes, respectively, and rotational components, θ_v and φ_v, in the yx and zy planes, respectively.



Virtual Work due to External Force Systems

If we consider a small element, δ_x, of the member at the point, the distributed load may be regarded as constant over the length δ_x and acting, in effect, as a concentrated load $w(x).\delta_x$ The virtual work, $w(x).\delta_x$, done by the complete external force system is therefore given by

$$w_e = W.\Delta_{v,y} + P.\Delta_{v,x} + M.\theta_v + T.\phi_v + \int_L w(x)\Delta_{v,y}.dx$$

Virtual Work due to External Force Systems

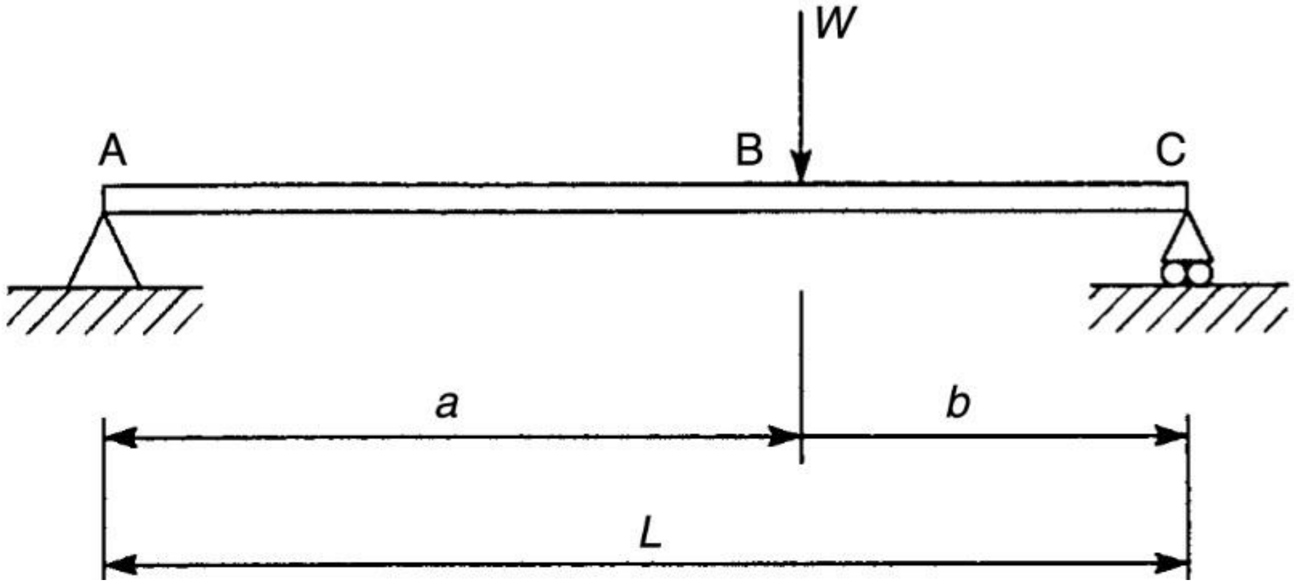
For a structure comprising a number of load positions, the total external virtual work done is then

$$w_e = \sum \left[W.\Delta_{v,y} + P.\Delta_{v,\mathrm{x}} + M.\theta_v + T.\phi_v + \int_L w(\mathrm{x})\Delta_{v,y}.dx \right]$$

We could, of course, have forces and moments and components of the virtual displacement in a horizontal plane, in which case this equation would be extended to include their contribution.

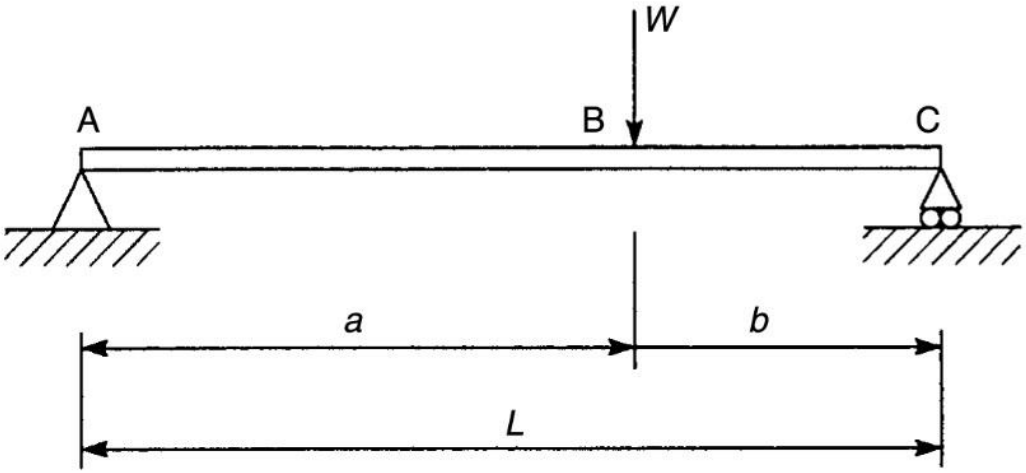
Example 1

Determine the bending moment at the point B in the simply supported beam ABC shown below

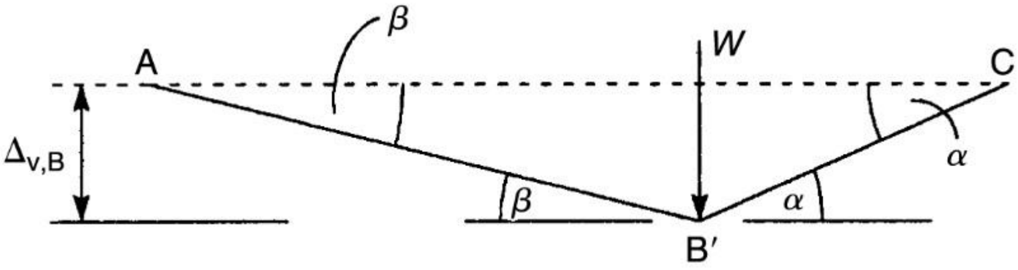


Example 1

$$M_B = \frac{W.a.b}{L}$$



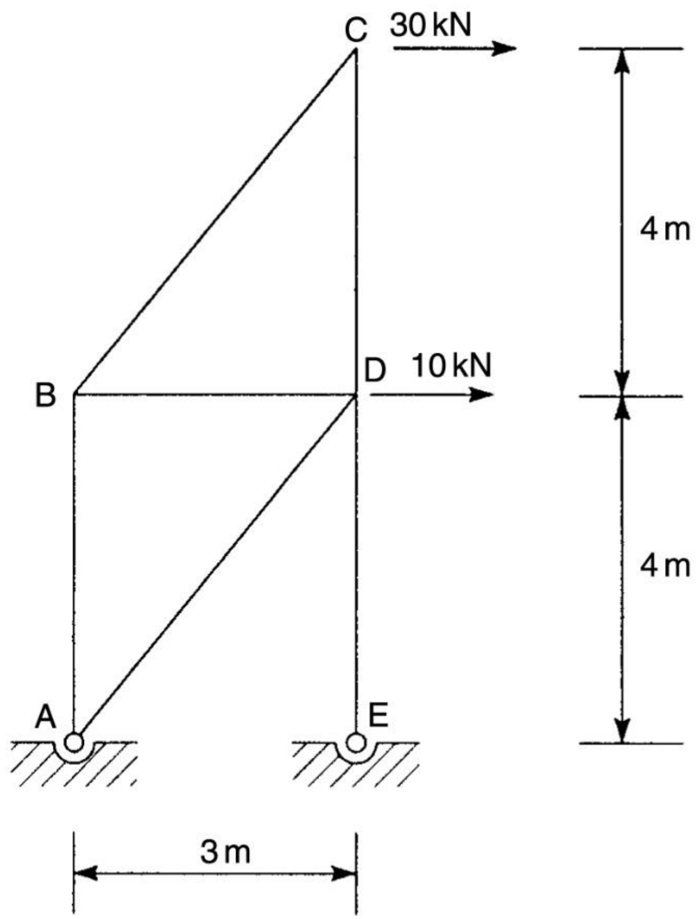
(a)



(b)

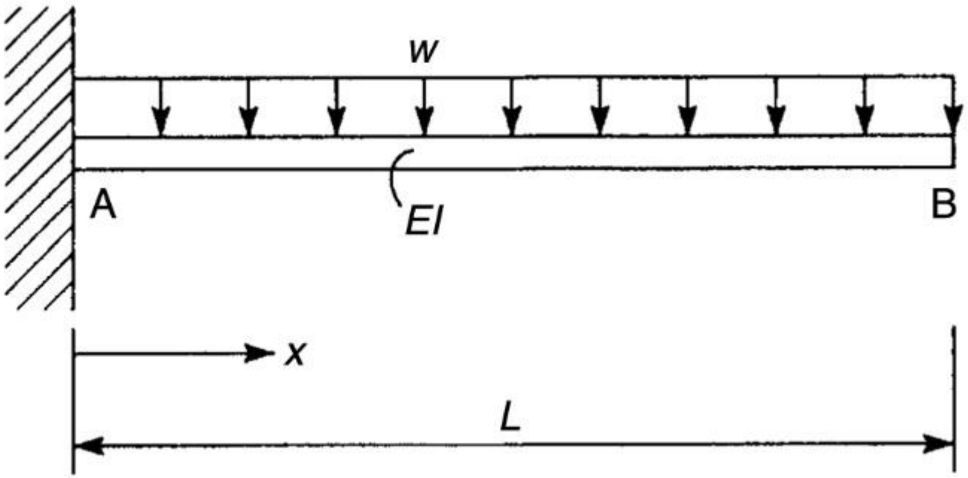
Example 2

Determine the force in the member AB in the truss shown below.



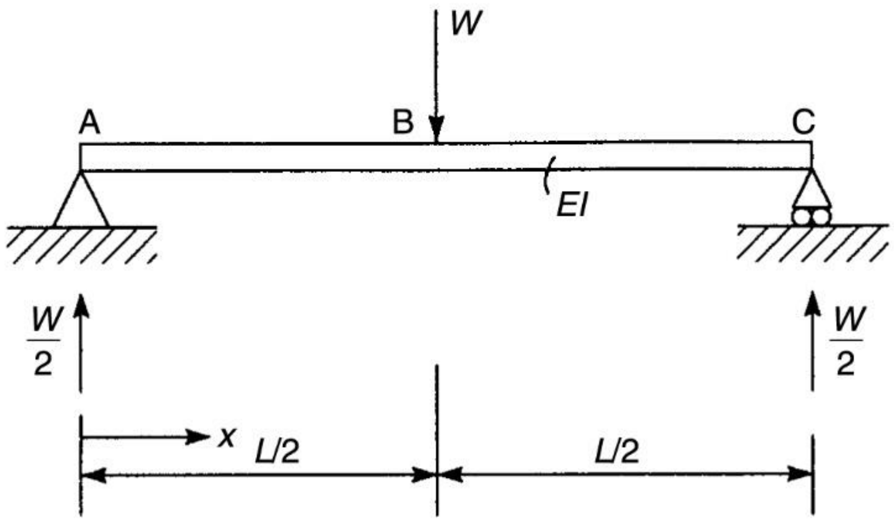
Example 3

Determine the vertical deflection of the free end of the cantilever beam shown below



Example 4

Determine the rotation, i.e. the slope, of the beam ABC shown



Example 3

Calculate the vertical deflection of the joint B and the horizontal movement of the support D in the truss shown.

The cross-sectional area of each member is 1800 mm2 and Young's modulus, E, for the material of the members is 200 000 N/mm2.