Aerospace Structural Analysis

Lecture 3

Strain Transformation

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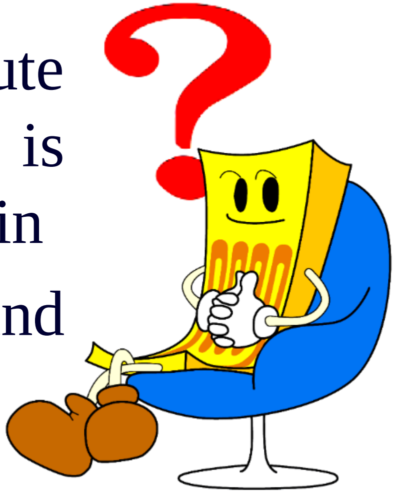
Lecture Objectives

- ✓ Derive equations for transforming strain components between coordinate systems of different orientation
- ✓ Use derived equations to obtain the maximum normal and maximum shear strain at a point
- ✓ Determine the orientation of elements upon which the maximum normal and maximum shear strain acts
- ✓ Discuss a method for determining the absolute maximum shear strain at a point when material is subjected to plane and 3-dimensional states of strain
- ✓ Derive equations for determining the strain and stress using strain rosettes.



Lecture Outline

- ✓ Plane-Strain Transformation
- ✓ General Equations of Plane Strain Transformation
- ✓ Principal strain and Maximum In-Plane Shear Strain
- ✓ Mohr's Circle – Plane Strain
- ✓ Absolute Maximum Shear strain
- ✓ Strain Gauge
- ✓ Strain Rosettes.

Plane-Strain Vs Plane-Stress

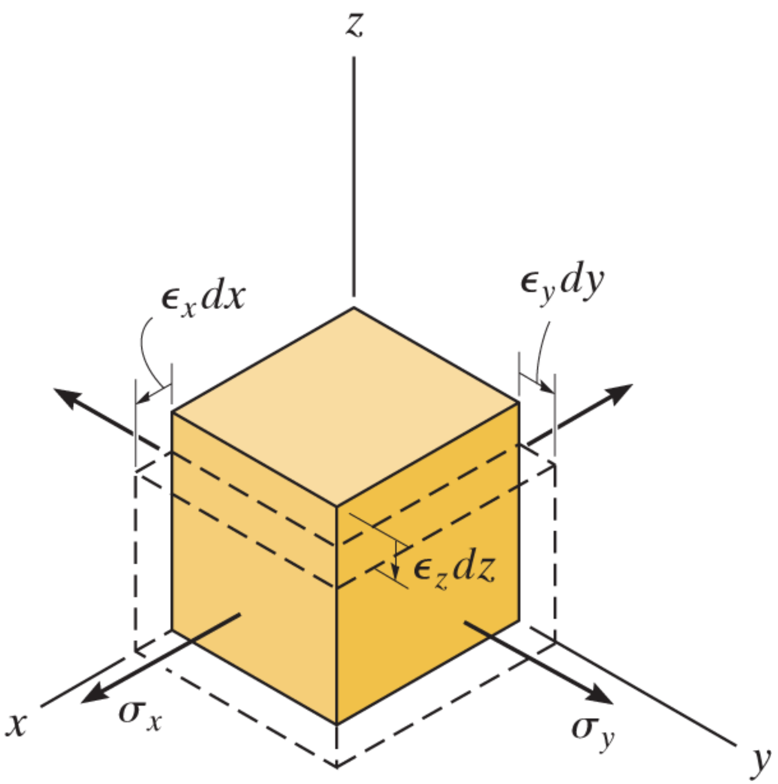
Plane Stress	The geometry of the body is essentially that of a plate with one dimension much smaller than the others.
Plane strain	the dimension of the structure in one direction, is very large in comparison with the dimensions of the structure in the other two directions

Plane-Strain Vs Plane-Stress

Plane Stress	Components σ_z, τ_{yz} , and τ_{xz} are neglected Components σ_x, σ_y and τ_{yx} are developed in the element
Plane strain	Components ε_z, γ_{yz} , and γ_{xz} are neglected Components ε_x, ε_y and γ_{yx} are developed in the element

Plane-Strain Vs Plane-Stress

Although plane strain and plane stress each have three components lying in the same plane, realize that plane stress does not necessarily cause plane strain or vice versa.



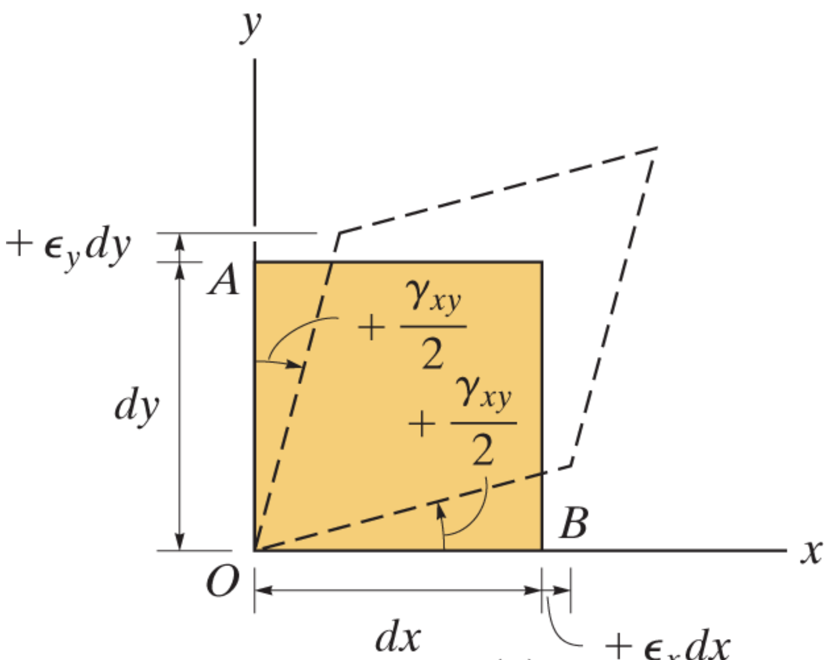
Plane stress

Plane-Strain Transformation

Sign Convention.

Normal strains ε_x and ε_y are positive if they cause elongation along the x and y axes, respectively.

The shear strain γ_{yx} is positive if the interior angle AOB becomes smaller than 90°. This sign convention also follows the corresponding one used for plane stress.



(a)

Plane-Strain

If the angle between the x and x' axes is θ

$$\varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\frac{\gamma_{x'y'}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

$$\varepsilon_{y'} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta$$

Principal In-Plane Strains

It can be seen that the magnitudes of $\varepsilon_{x'}$, $\varepsilon_{y'}$ and $\gamma_{x'y'}$ depend on the angle of inclination θ of the planes on which these deformation measured. In engineering practice it is often important to determine the orientation of the element that element's deformation is caused only by normal strains, with no shear strain. When this occurs the normal strains are referred to as principal strains, and if the material is isotropic, the axes along which these strains occur will coincide with the axes that define the planes of principal stress.

Principal In-Plane Strains

$$\tan 2\theta_P = \frac{\gamma_{xy}}{\left(\varepsilon_x - \varepsilon_y\right)}$$

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\frac{\gamma}{2} = 0$$

Maximum In-Plane Shear Strain

$$\tan 2\theta_s = -\frac{\left(\varepsilon_x - \varepsilon_y\right)}{\gamma_{xy}}$$

$$\frac{\gamma_{\substack{\max \\ \text{in-plane}}}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\varepsilon_{x'} = \varepsilon_{y'} = \varepsilon_{avg} = \frac{\varepsilon_x + \varepsilon_y}{2}$$

Mohr’s Circle Plane Strain

$$\varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\frac{\gamma_{x'y'}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

$$\varepsilon_{x'} - \frac{\varepsilon_x + \varepsilon_y}{2} = +\frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\frac{\gamma_{x'y'}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

$$\left(\varepsilon_{x'} - \frac{\varepsilon_x + \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{x'y'}}{2}\right)^2 = \left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2$$

Mohr's Circle Plane Strain

$$\left(\varepsilon_{x'} - \frac{\varepsilon_x + \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{x'y'}}{2}\right)^2 = \left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2$$

$$\left(\varepsilon_{x'} - \varepsilon_{avg}\right)^2 + \left(\frac{\gamma_{x'y'}}{2}\right)^2 = R^2 \quad \begin{cases} \varepsilon_{avg} = \dfrac{\varepsilon_x + \varepsilon_y}{2} \\ R = \sqrt{\left(\dfrac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\dfrac{\gamma_{xy}}{2}\right)^2} \end{cases}$$

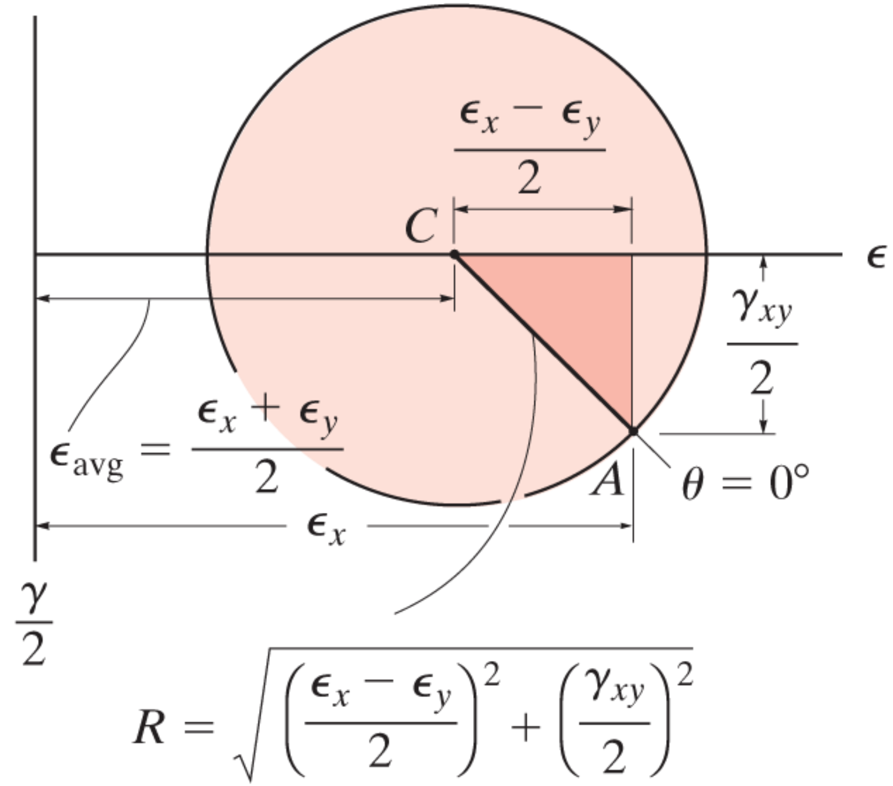
Mohr's Circle Plane Strain

$$\left(\varepsilon_{x'} - \frac{\varepsilon_x + \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{x'y'}}{2}\right)^2 = \left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2$$

$$\left(\varepsilon_{x'} - \varepsilon_{avg}\right)^2 + \left(\frac{\gamma_{x'y'}}{2}\right)^2 = R^2$$

$$\varepsilon_{avg} = \frac{\varepsilon_x + \varepsilon_y}{2}$$

$$R = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

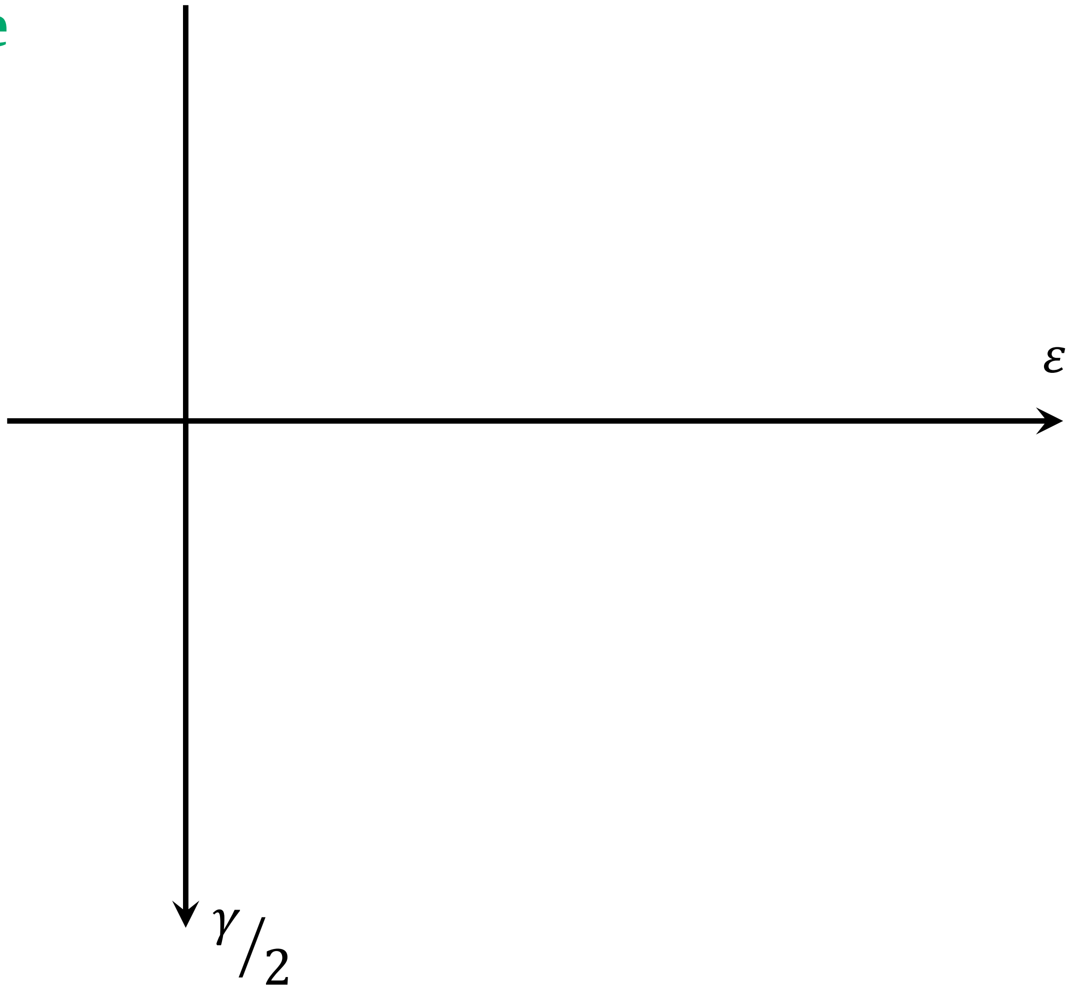


Mohr's Circle Plane Strain

Construction of the Circle

If we establish coordinate axes, ε positive to the right and $\gamma/2$ positive downward, and then plot the equation

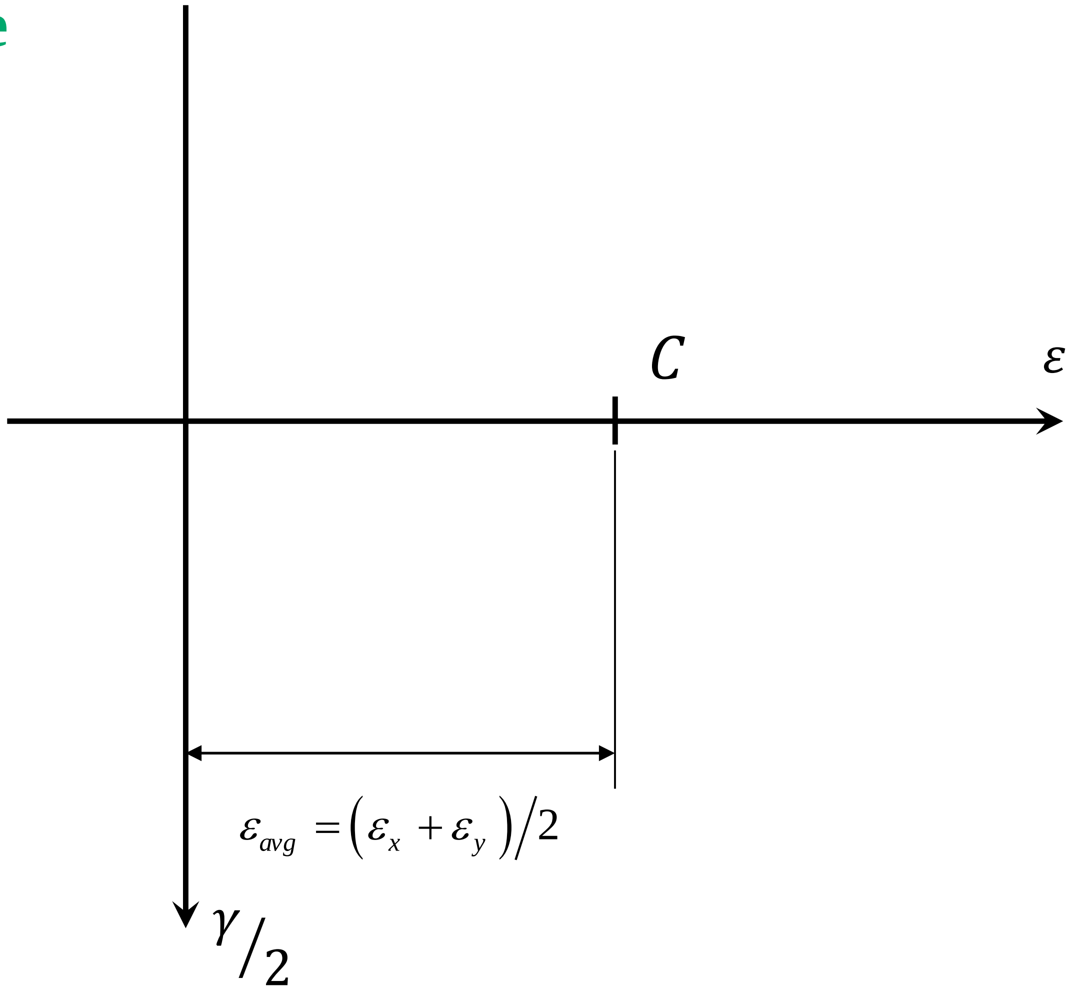
$$\left(\varepsilon_{x'} - \varepsilon_{avg}\right)^2 + \left(\frac{\gamma_{x'y'}}{2}\right)^2 = R^2$$



Mohr's Circle Plane Strain

Construction of the Circle

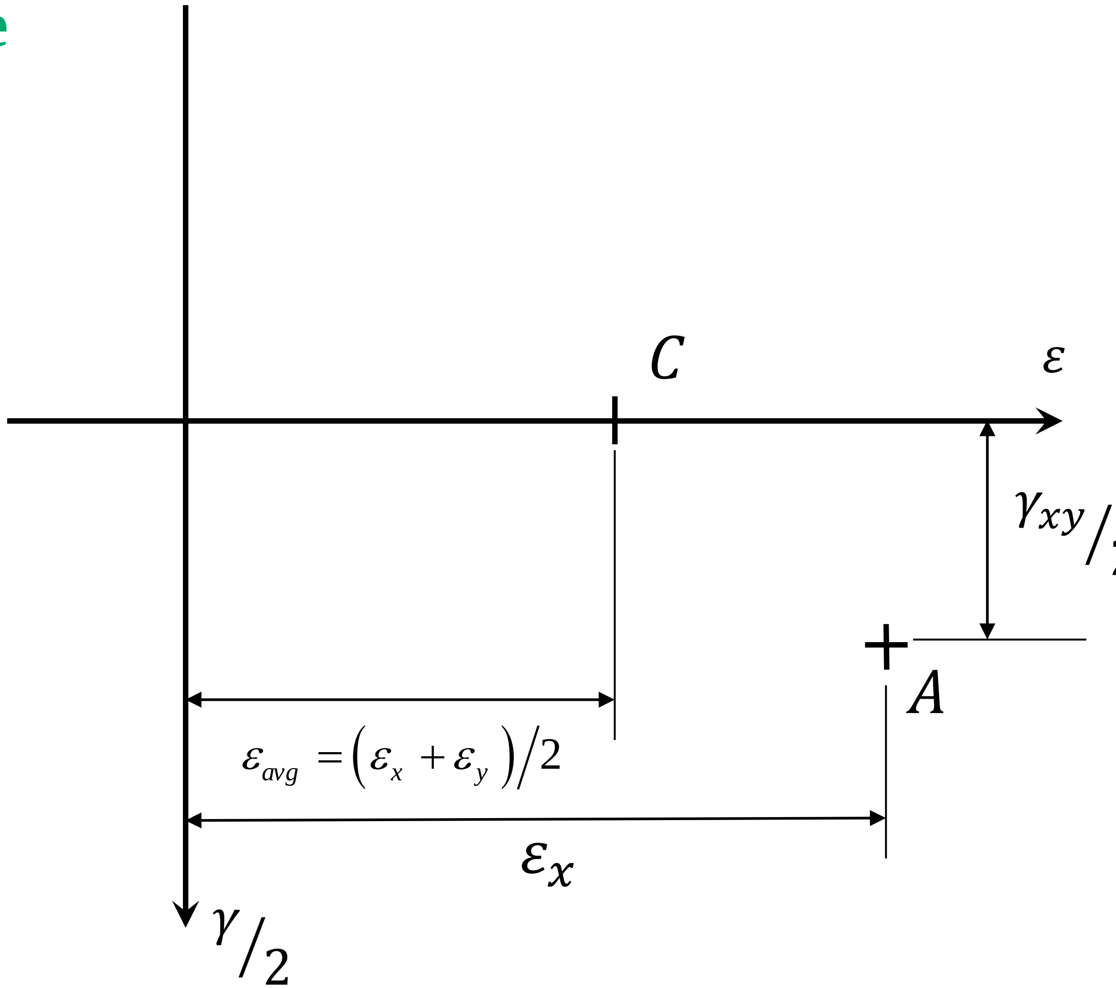
Using the positive sign convention for ε_x , ε_y and γ_{xy}, plot the center of the circle C, which is located on the axis at a distance C:$\left(\varepsilon_{avg}, 0\right)$ from the origin.



Mohr's Circle Plane Strain

Construction of the Circle

Plot the "reference point" A having coordinates $A(\varepsilon_x, {}^{\gamma_{xy}}/_{2})$.
This point represents the normal and shear strain components on the element's right-hand vertical face, and since the x axis coincides with the x' axis, this represents $\theta = 0$

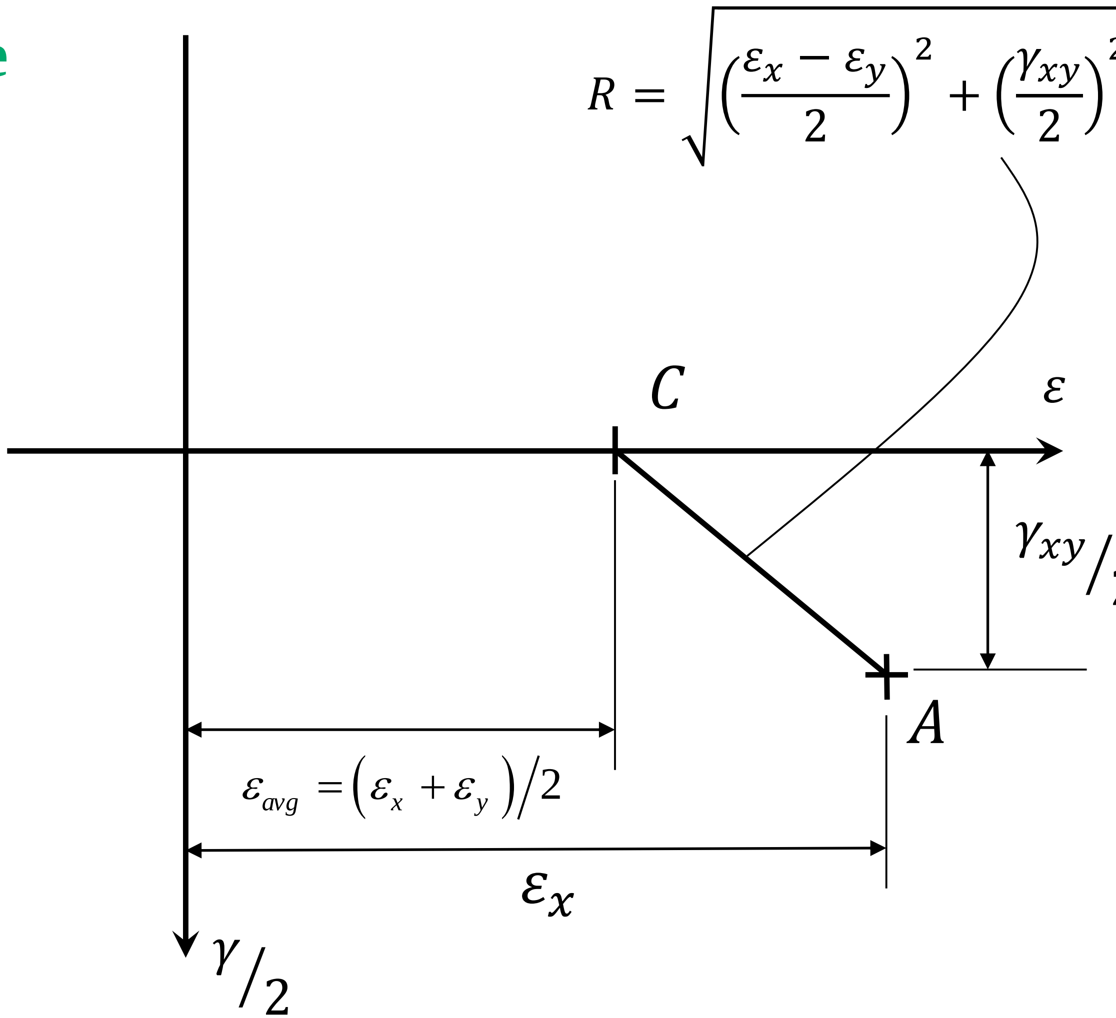


Mohr's Circle Plane Strain

Construction of the Circle

Connect point A with the center C of the circle and determine CA by trigonometry. This distance represents the radius R of the circle,

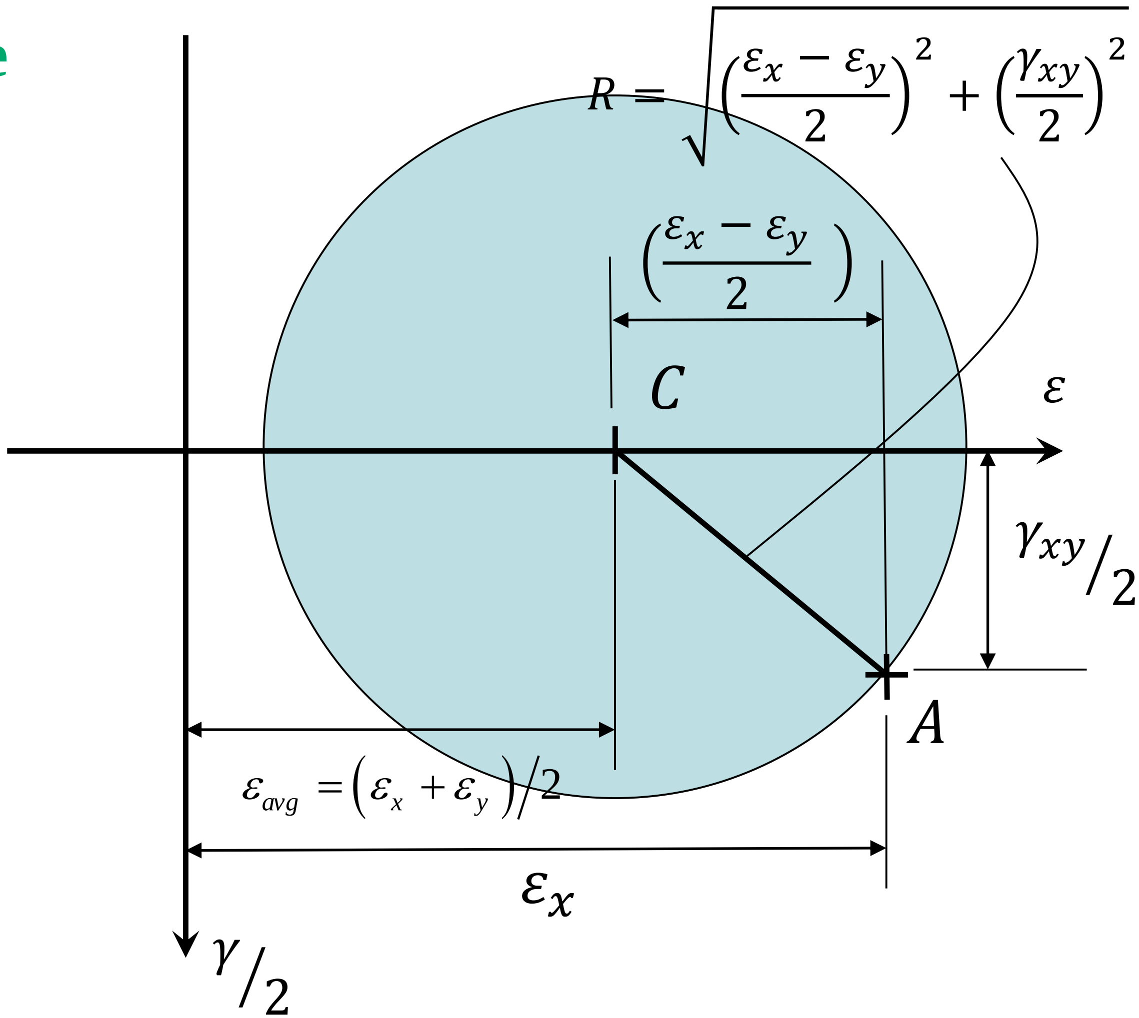
$$R = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$



Mohr's Circle Plane Strain

Construction of the Circle

Once R has been determined, sketch the circle

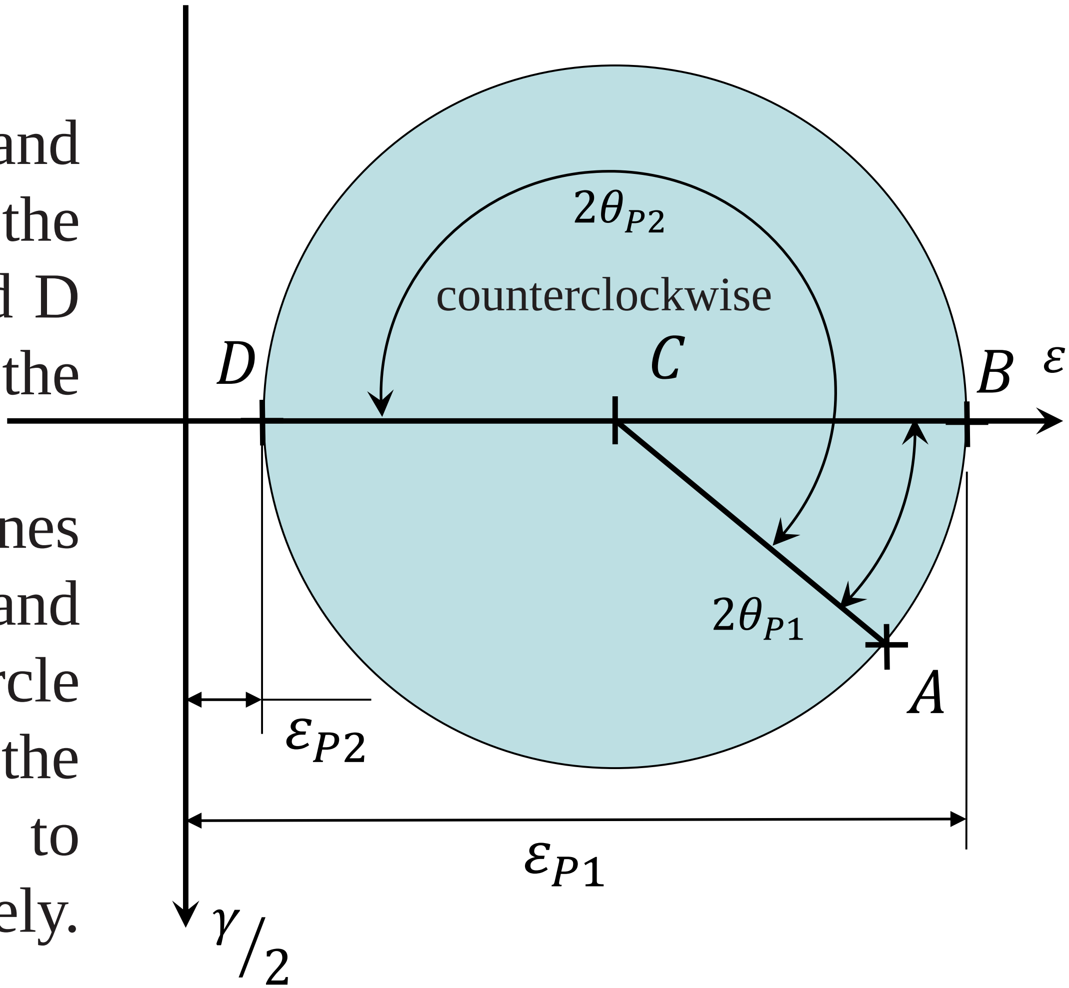


Mohr's Circle Plane Strain

Principal Strains

The principal strains ε_{P1} and ε_{P2} $(\varepsilon_{P1} \geq \varepsilon_{P2})$ are the coordinates of points B and D where the circle intersects the axis ε, i.e., where $\gamma = 0$

These stresses act on planes defined by angles θ_{P1} and θ_{P2}, represented on the circle by angles measured from the radial reference line CA to lines CB and CD, respectively.

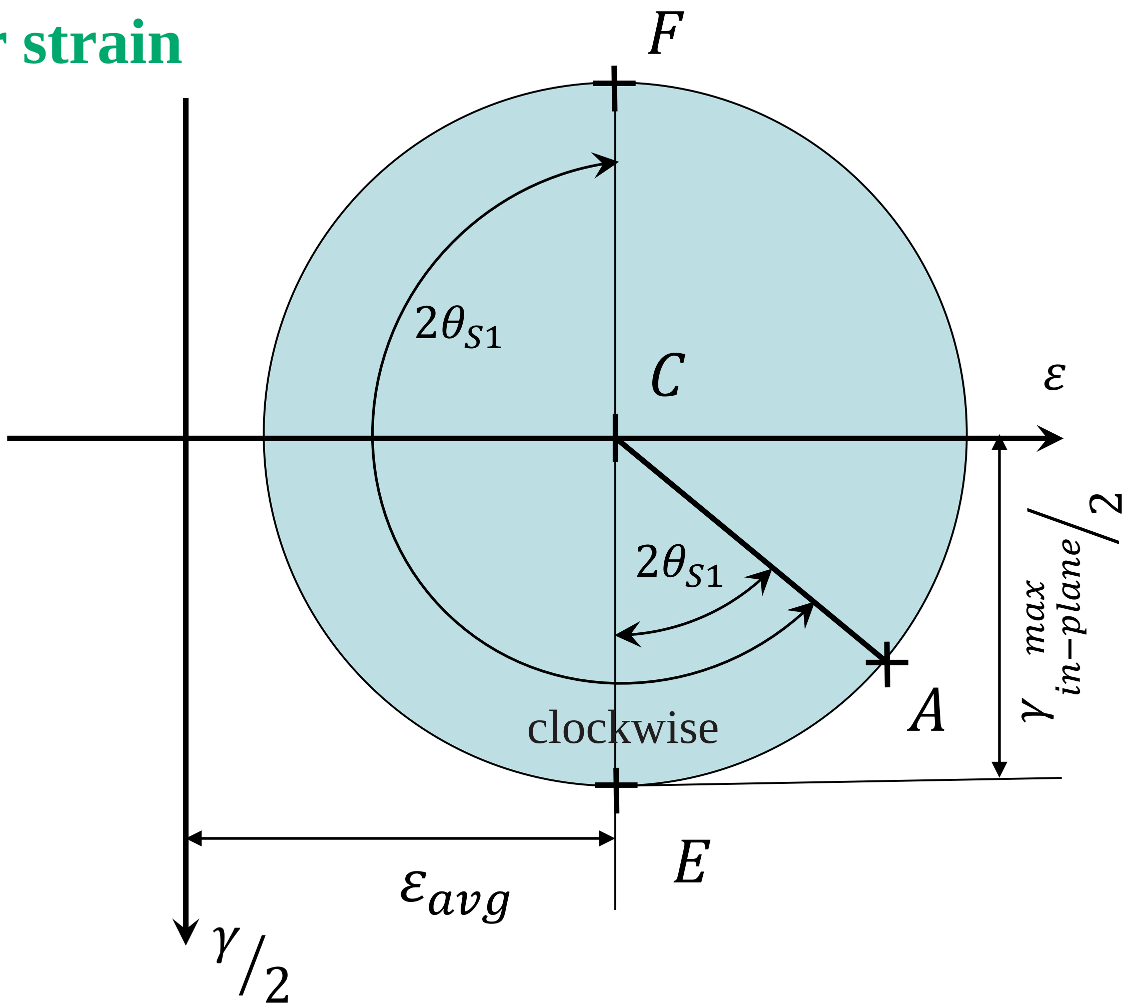


Mohr's Circle Plane Strain

Maximum In-Plane Shear strain

The average normal strain and maximum in-plane shear strain components are determined from the circle as the coordinates of either point E or F.

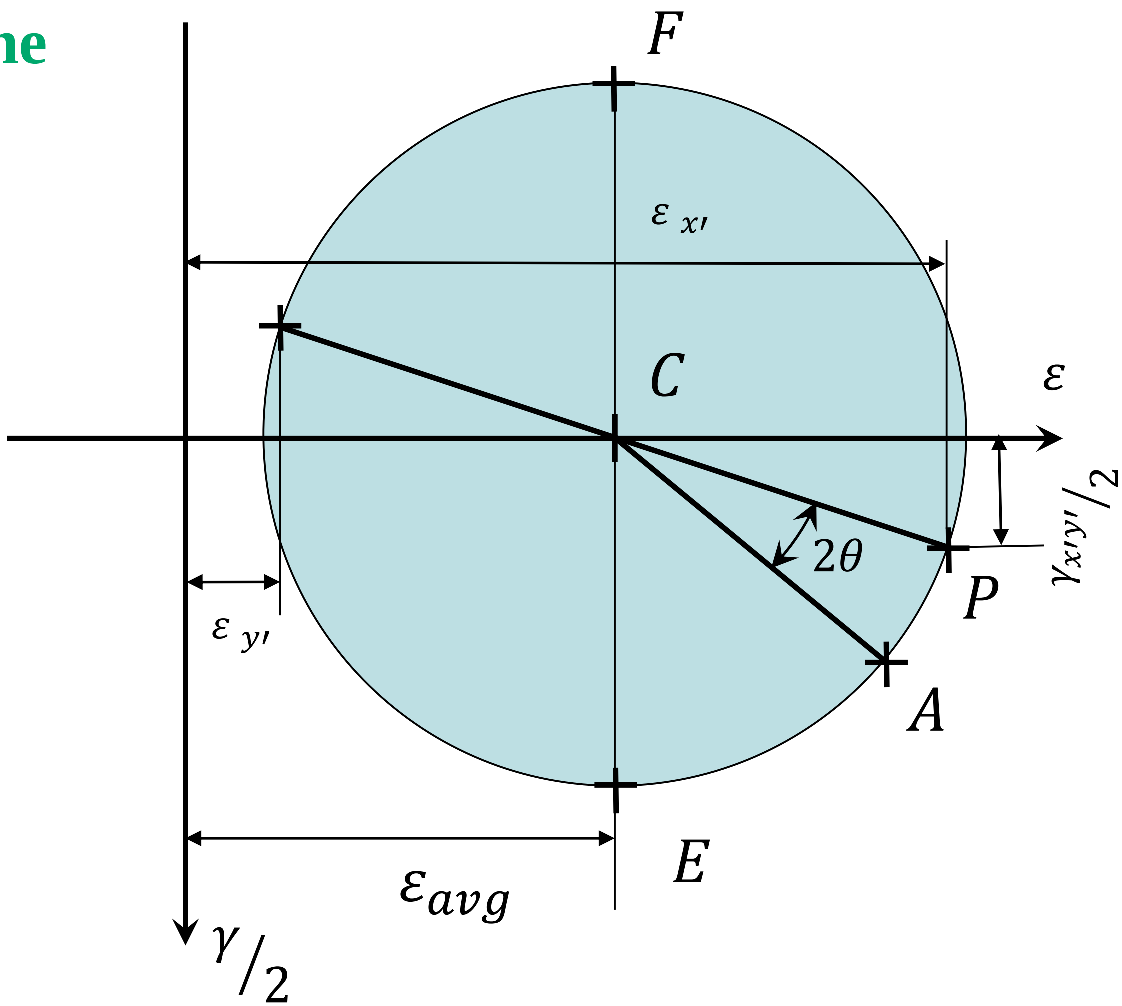
the angles and give the orientation of the planes that contain these components.



Mohr's Circle Plane Strain

Stresses on Arbitrary Plane

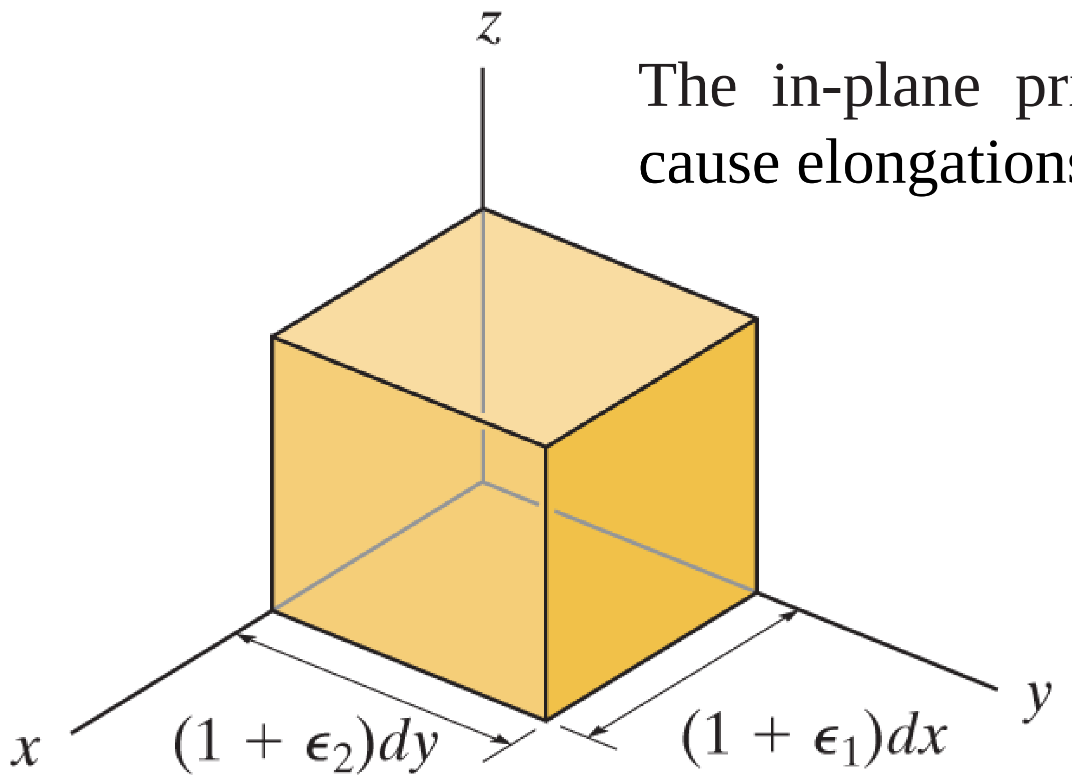
To locate P, the known angle θ (in this case counterclockwise), must be measured on the circle in the same direction 2θ, from the radial reference line CA to the radial line CP



Absolute maximum shear strain

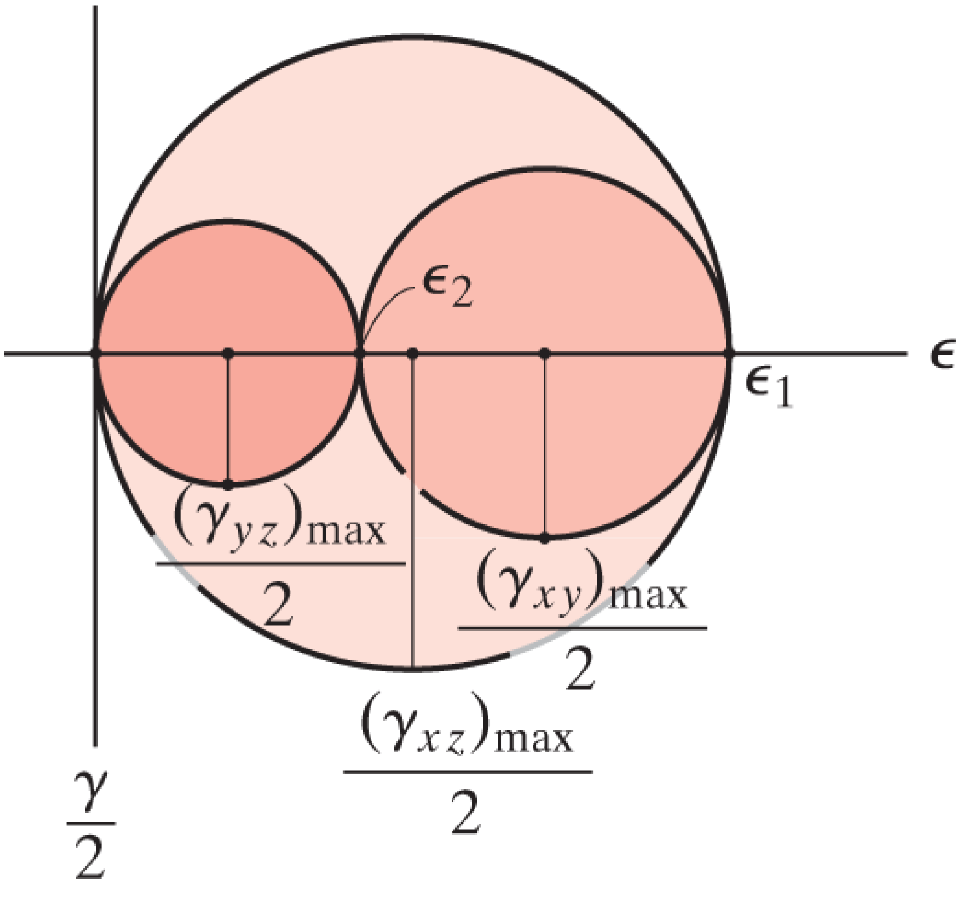
ε_1 and ε_2 have the same sign

The in-plane principal strains ε_1 and ε_2 cause elongations



$x - y$ plane strain

Absolute maximum shear strain



ε_1 and ε_2 have the same sign

For all three circles, it is seen that although the maximum in-plane shear strain is

$$\gamma_{x'y'} = (\varepsilon_1 - \varepsilon_2)$$

This value is *not* the absolute maximum shear strain. Instead, from the figure

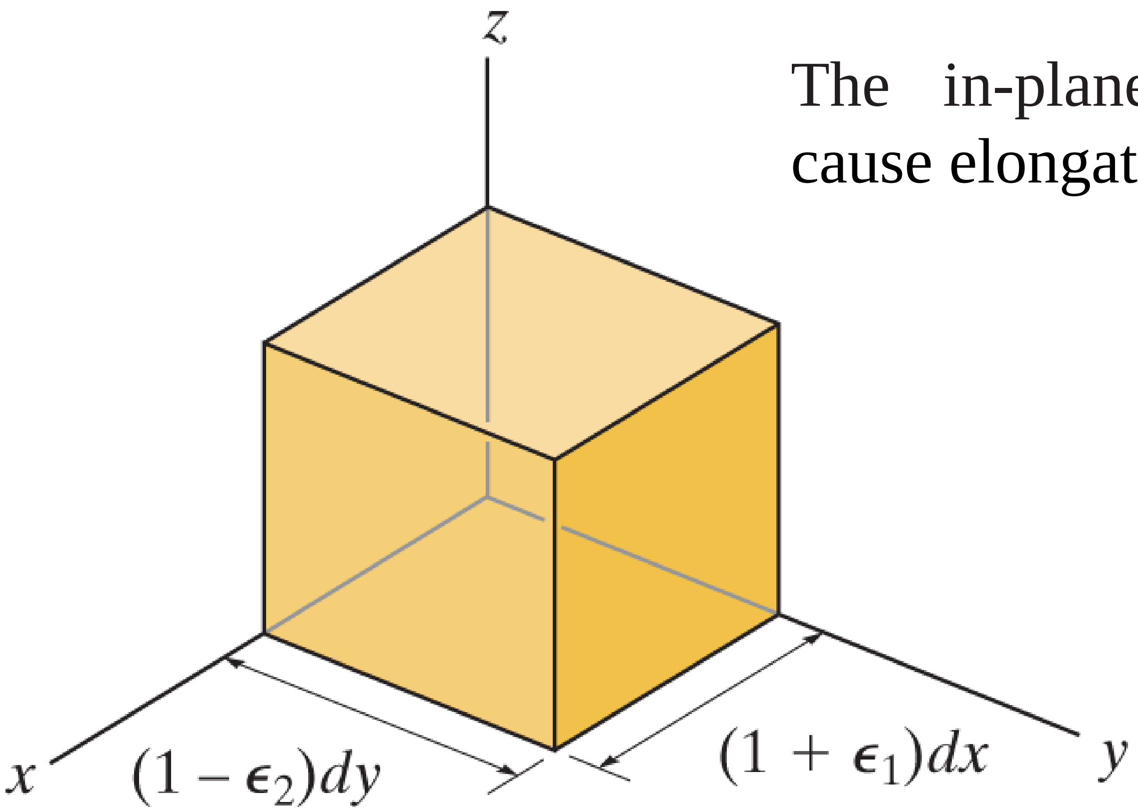
$$\gamma_{\substack{abs\\max}} = \varepsilon_1$$

The absolute maximum shear strain will occur out of the plane

Absolute maximum shear strain

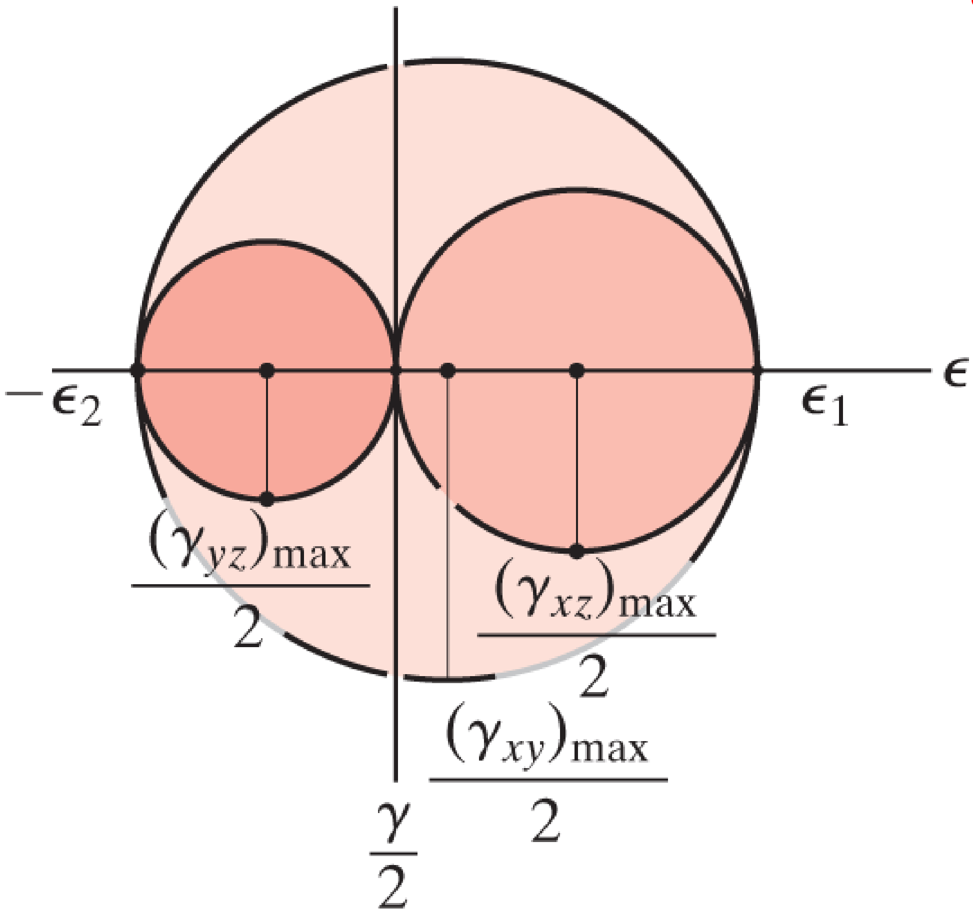
ε_1 and ε_2 have the opposite sign

The in-plane principal strains ε_1 cause elongations and ε_2 contraction.



$x-y$ plane strain

Absolute maximum shear strain



ε_1 and ε_2 have the opposite signs

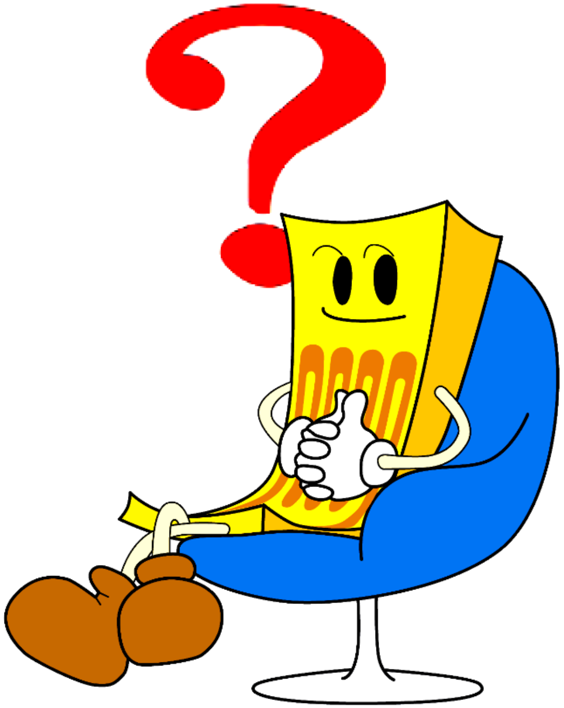
For all three circles, it is seen that the maximum in-plane shear strain is equal to the absolute maximum shear stress:

$$\gamma_{\substack{max \\ abs}} = \gamma_{x'y'} = \varepsilon_1 - \varepsilon_2$$

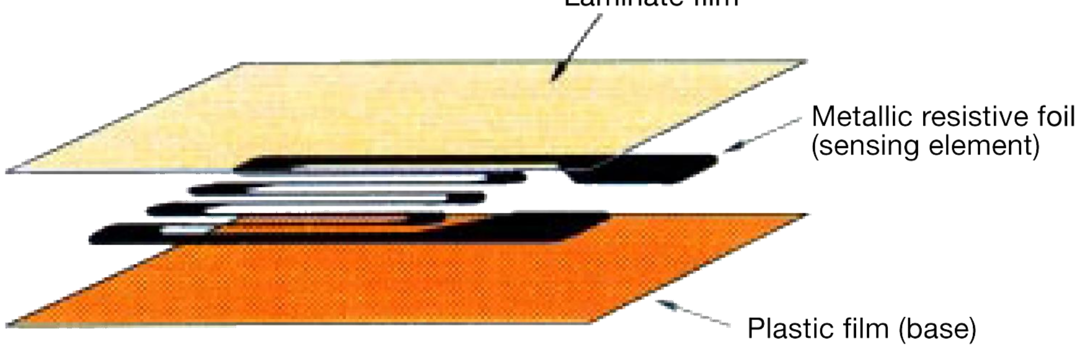
Strain Gauge

What is a Strain Gauge ?

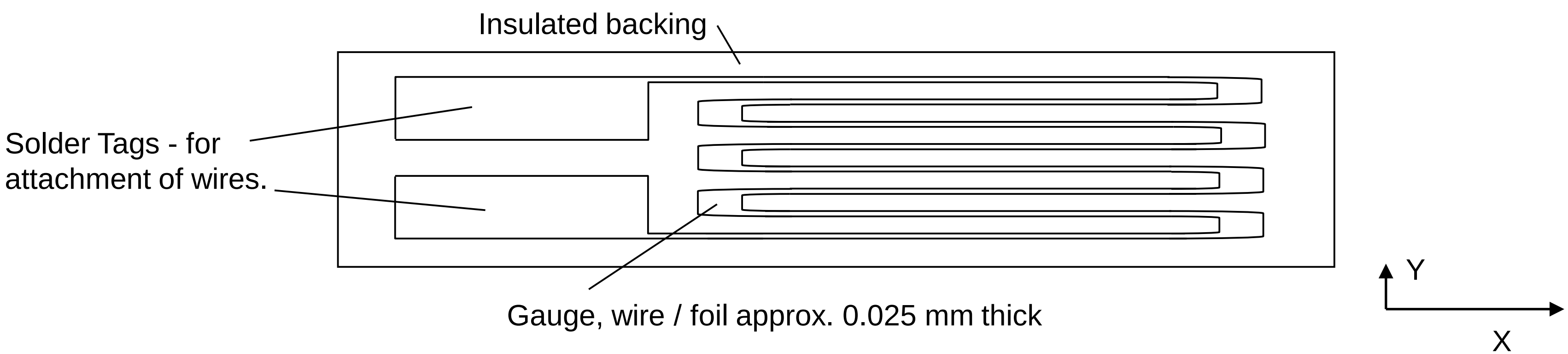
1. Strain Gauge is a device used to measure deformation (strain) of an object.
2. Strain gauges have been developed for the accurate measurement of strain
3. Fundamentally, all strain gauges are designed to convert mechanical motion into an electronic signal.



Strain Gauge



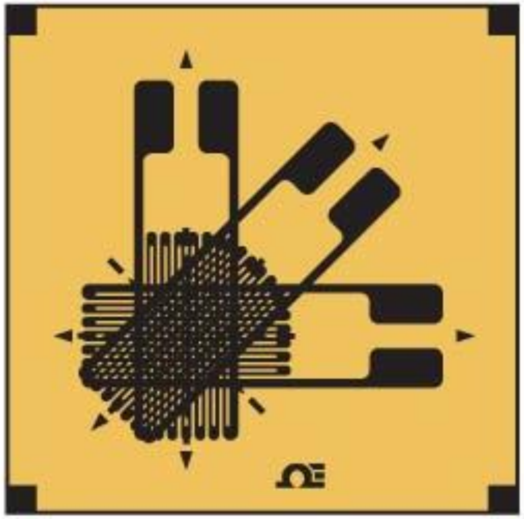
Schematic View of Strain Gauge

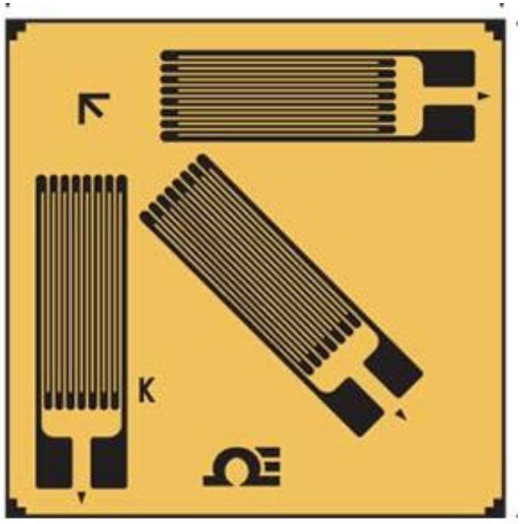


The gauge shown here is primarily sensitive to strain in the X direction, as the majority of the wire length is parallel to the X axis.

Strain Rosettes

For a general loading on a body, however, the strains at a point on its free surface are determined using a cluster of three electrical-resistance strain gauges, arranged in a specified pattern.





This pattern is referred to as a **strain rosette**, and once the normal strains on the three gauges are measured, the data can then be transformed to specify the state of strain at the point.

Strain Rosettes

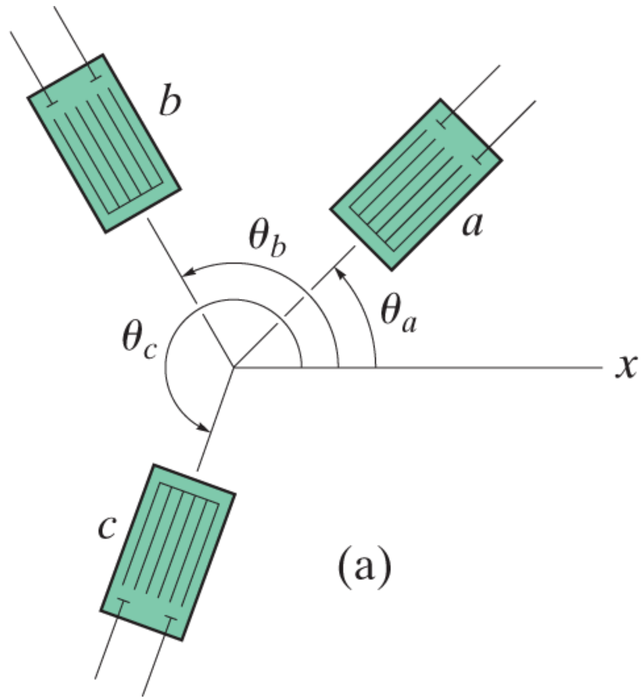
Since these strains are measured only in the plane of the gauges, and since the body is stress-free on its surface, the gauges may be subjected to plane stress but not plane strain.

Although the strain normal to the surface is not measured, realize that the out-of-plane displacement caused by this strain will not affect the in-plane measurements of the gauges.

Strain Rosettes

In the general case, the axes of the three gauges are arranged at the angles θ_a, θ_b and θ_c.

If the readings ε_a, ε_b and ε_c are taken, we can determine the strain components ε_x, ε_y and γ_{xy} at the point by applying the strain-transformation equation, for each gauge. We have



$$\varepsilon_a = \varepsilon_x \cos^2\theta_a + \varepsilon_y \sin^2\theta_a + \gamma_{xy} \sin\theta_a \cos\theta_a$$

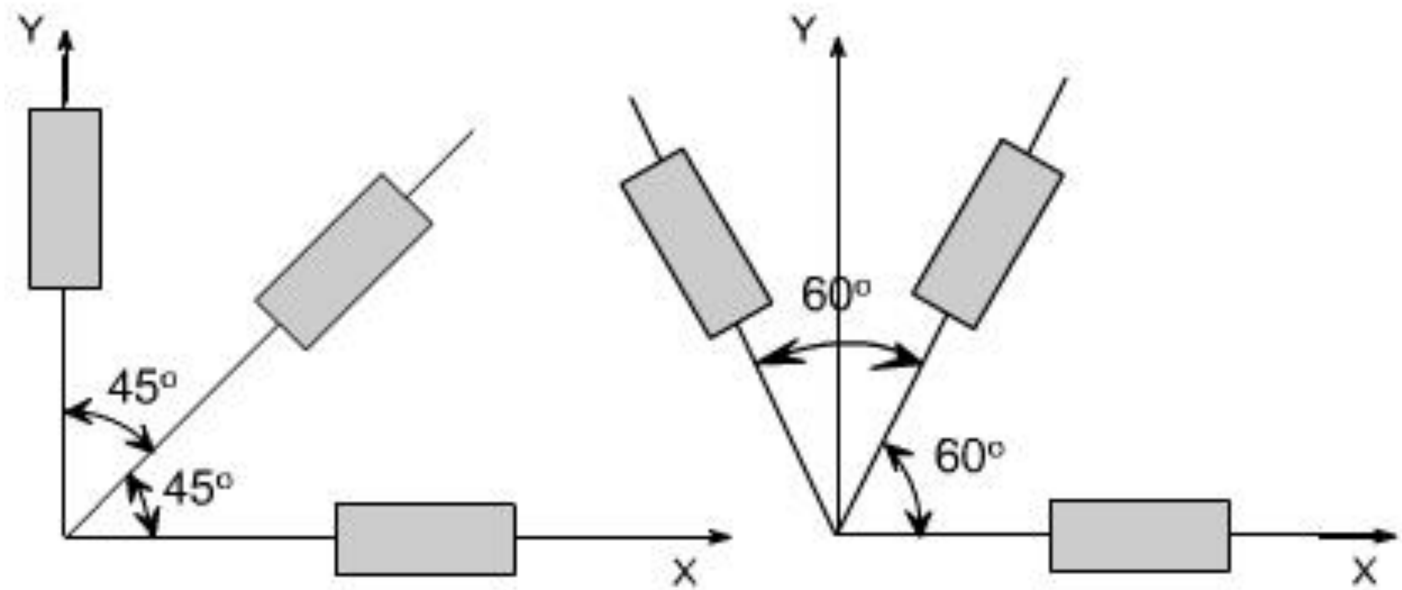
$$\varepsilon_b = \varepsilon_x \cos^2\theta_b + \varepsilon_y \sin^2\theta_b + \gamma_{xy} \sin\theta_b \cos\theta_b$$

$$\varepsilon_c = \varepsilon_x \cos^2\theta_c + \varepsilon_y \sin^2\theta_c + \gamma_{xy} \sin\theta_c \cos\theta_c$$

Strain Rosettes

The values of ε_x, ε_y and γ_{xy} are determined by solving these three equations simultaneously.

Strain rosettes are often arranged in 45° or 60° patterns.



0°/45°/90° Rosette

In the case of the 45° or "rectangular", θ_a=0°, θ_b=45° and θ_c =90°.

$$\varepsilon_x = \varepsilon_a$$

$$\varepsilon_y = \varepsilon_c$$

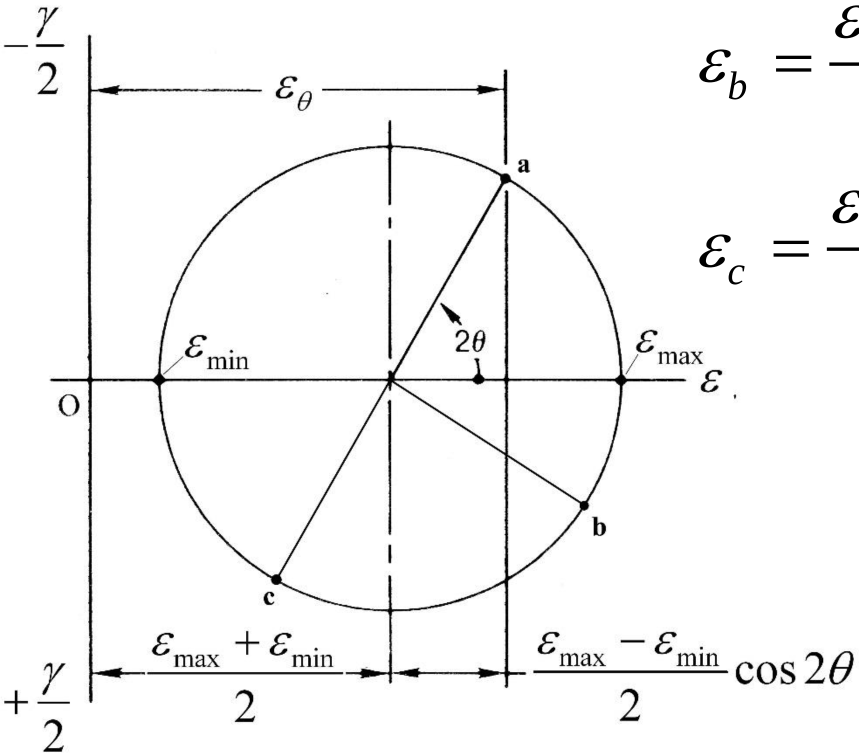
$$\gamma_{xy} = 2.\varepsilon_b - \left(\varepsilon_a + \varepsilon_c\right)$$

0°/45°/90° Rosette

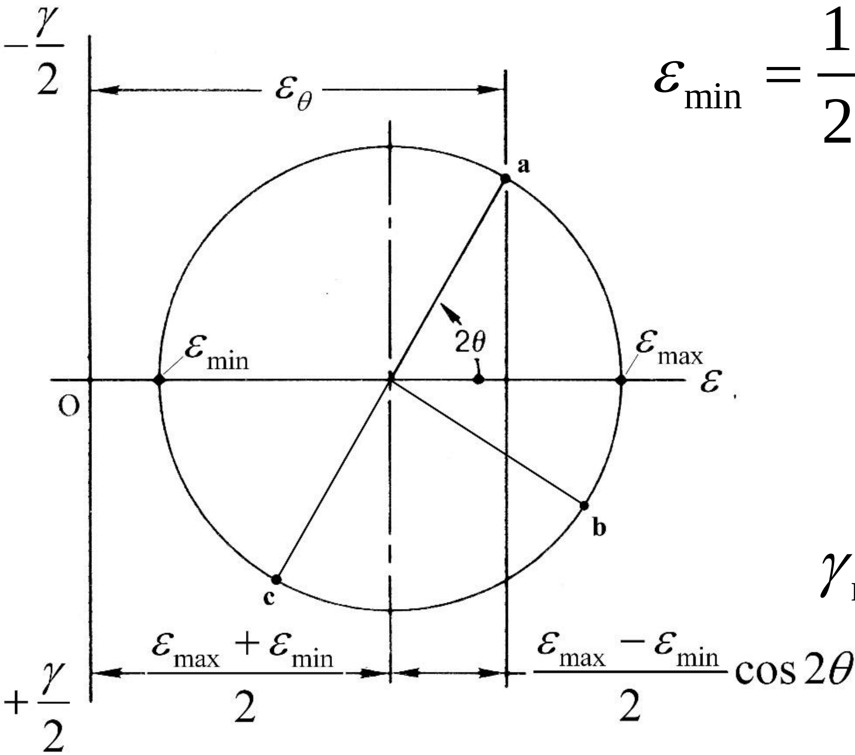
$$\varepsilon_a = \frac{\varepsilon_{max} + \varepsilon_{min}}{2} + \frac{\varepsilon_{max} - \varepsilon_{min}}{2} \cos 2\theta$$

$$\varepsilon_b = \frac{\varepsilon_{max} + \varepsilon_{min}}{2} + \frac{\varepsilon_{max} - \varepsilon_{min}}{2} \cos 2(45° - \theta)$$

$$\varepsilon_c = \frac{\varepsilon_{max} + \varepsilon_{min}}{2} + \frac{\varepsilon_{max} - \varepsilon_{min}}{2} \cos 2(90° - \theta)$$



0°/45°/90° Rosette



$$\varepsilon_{max} = \frac{1}{2}\left[\varepsilon_a + \varepsilon_c + \sqrt{2\left\{(\varepsilon_a - \varepsilon_b)^2 + (\varepsilon_b - \varepsilon_c)^2\right\}}\right]$$

$$\varepsilon_{min} = \frac{1}{2}\left[\varepsilon_a + \varepsilon_c - \sqrt{2\left\{(\varepsilon_a - \varepsilon_b)^2 + (\varepsilon_b - \varepsilon_c)^2\right\}}\right]$$

$$\theta = \frac{1}{2}\tan^{-1}\left[\frac{2\varepsilon_b - \varepsilon_a - \varepsilon_c}{\varepsilon_a - \varepsilon_c}\right]$$

$$\gamma_{max} = \sqrt{2\left\{(\varepsilon_a - \varepsilon_b)^2 + (\varepsilon_b - \varepsilon_c)^2\right\}}$$

0°/45°/90° Rosette

$$\sigma_{1,2} = \frac{E}{1-\upsilon}\frac{\varepsilon_a + \varepsilon_c}{2} \pm \frac{E}{\sqrt{2}\left(1+\upsilon\right)}\sqrt{\left(\varepsilon_a - \varepsilon_b\right)^2 + \left(\varepsilon_c - \varepsilon_b\right)^2}$$

$$\theta = \tan^{-1}\left[\frac{2\varepsilon_b - \varepsilon_a - \varepsilon_c}{\varepsilon_a - \varepsilon_c}\right]$$

0°/45°/90° Rosette

$$\sigma_{\max} = \frac{E}{1-\nu^2}\left(\varepsilon_{\max} + \nu\varepsilon_{\min}\right)$$

$$\sigma_{\min} = \frac{E}{1-\nu^2}\left(\varepsilon_{\min} + \upsilon\varepsilon_{\max}\right)$$

0°/60°/120° Rosette

In the case of the 60°, θ_a=0°, θ_b=60° and θ_c =120°.

$$\varepsilon_x = \varepsilon_a$$

$$\varepsilon_y = \frac{1}{3}\left(2\varepsilon_b + 2\varepsilon_c - \varepsilon_a\right)$$

$$\gamma_{xy} = \frac{2}{\sqrt{3}}\left(\varepsilon_b - \varepsilon_c\right)$$

0°/60°/120° Rosette

$$\sigma_{1,2} = \frac{E}{1-\upsilon}\frac{\varepsilon_a + \varepsilon_b + \varepsilon_c}{3} \pm \frac{E}{1+\upsilon}\sqrt{\left(\frac{2\varepsilon_a - \varepsilon_b - \varepsilon_c}{3}\right)^2 + \frac{1}{3}\left(\varepsilon_c - \varepsilon_b\right)^2}$$

$$\theta = \tan^{-1}\left[\frac{\sqrt{3}\left(\varepsilon_b - \varepsilon_c\right)}{2\varepsilon_a - \varepsilon_b - \varepsilon_c}\right]$$

Material-Property Relationships

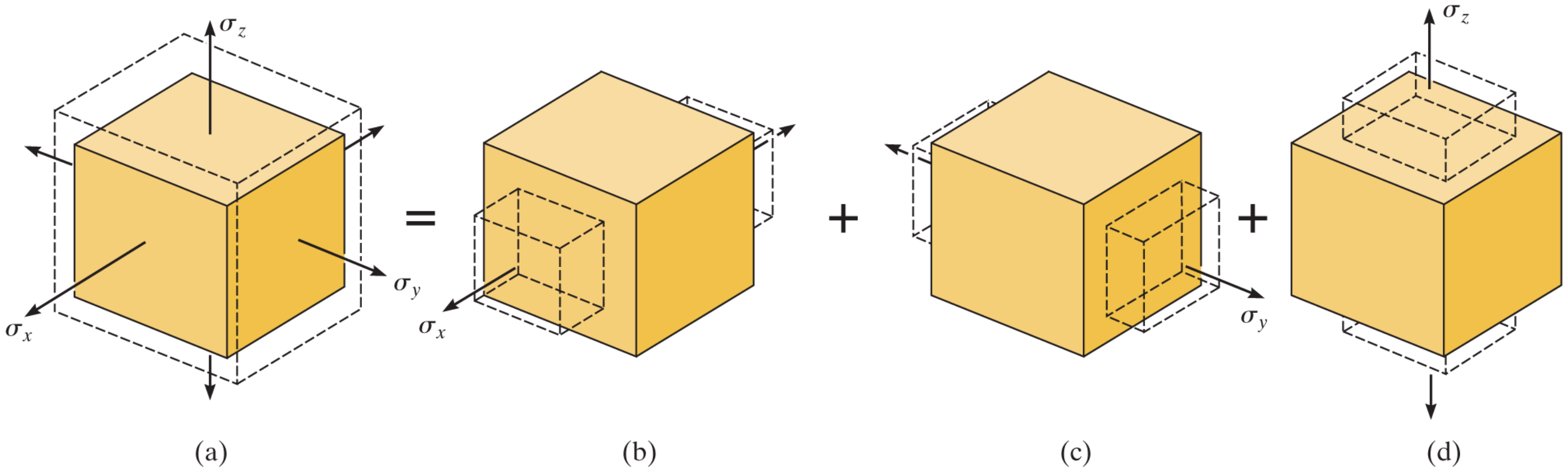
Assume that the material is homogeneous and isotropic and behaves in a linear-elastic manner

Material is subjected to multiaxial stress and strain.

Generalized Hooke's Law.

The stresses can be related to these strains by using the principle of superposition, Poisson's ratio, and Hooke's law.

Material-Property Relationships



$$\varepsilon_x \qquad \varepsilon'_x = \frac{\sigma_x}{E} \qquad \varepsilon''_x = -\nu\frac{\sigma_y}{E} \qquad \varepsilon'''_x = -\nu\frac{\sigma_z}{E}$$

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right)$$

Material-Property Relationships

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right)$$

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right)$$

$$\varepsilon_z = \frac{1}{E}\left(\sigma_z - \nu\left(\sigma_y + \sigma_x\right)\right)$$