Lecture 2

Stress Transformation

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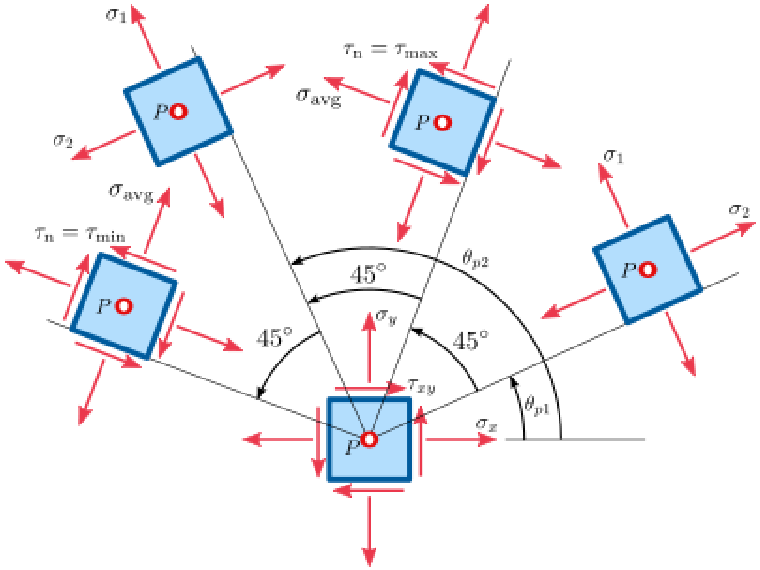
Lecture Objectives

- ✓ Derive equations for transforming stress components between coordinate systems of different orientation
- ✓ Use derived equations to obtain the maximum normal and maximum shear stress at a point



- ✓ Determine the orientation of elements upon which the maximum normal and maximum shear stress acts
- ✓ Discuss a method for determining the absolute maximum shear stress at a point when material is subjected to plane and 3-dimensional states of stress

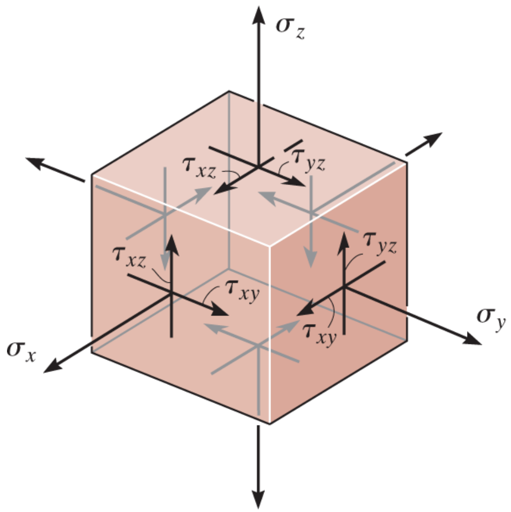
Lecture Outline

- ✓ Plane-Stress Transformation
- ✓ General Equations of Plane Stress Transformation
- ✓ Principal Stresses and Maximum In-Plane Shear Stress
- ✓ Mohr’s Circle – Plane Stress
- ✓ Absolute Maximum Shear Stress

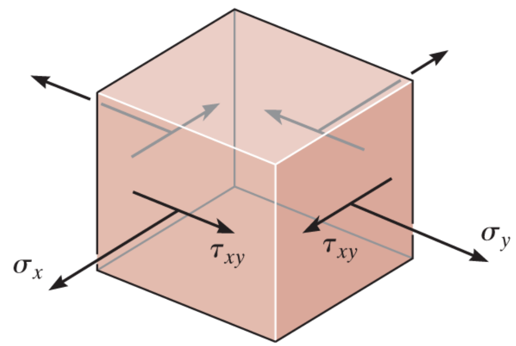
Plane-Stress Transformation

General state of stress at a point is characterized by six independent normal and shear stress components.

In practice, approximations and simplifications are done to reduce the stress components to a single plane.



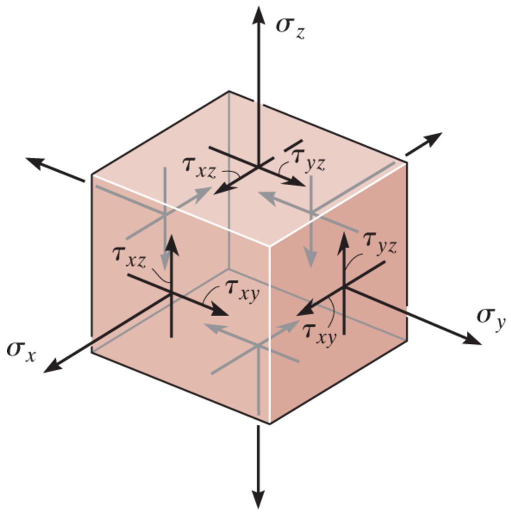
General state of stress



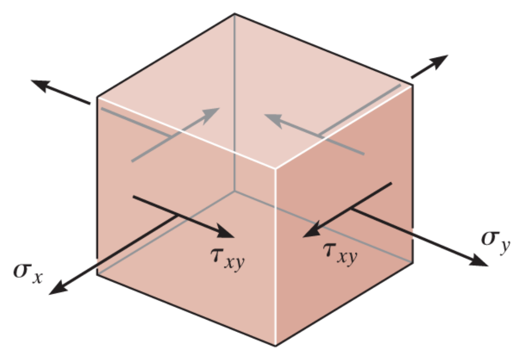
Plane stress

Plane-Stress Transformation

There is no load on the surface of a body, then the normal and shear stress components will be zero on the face of an element that lies on this surface. Consequently, the corresponding stress components on the opposite face will also be zero, and so the material at the point will be subjected to plane stress.



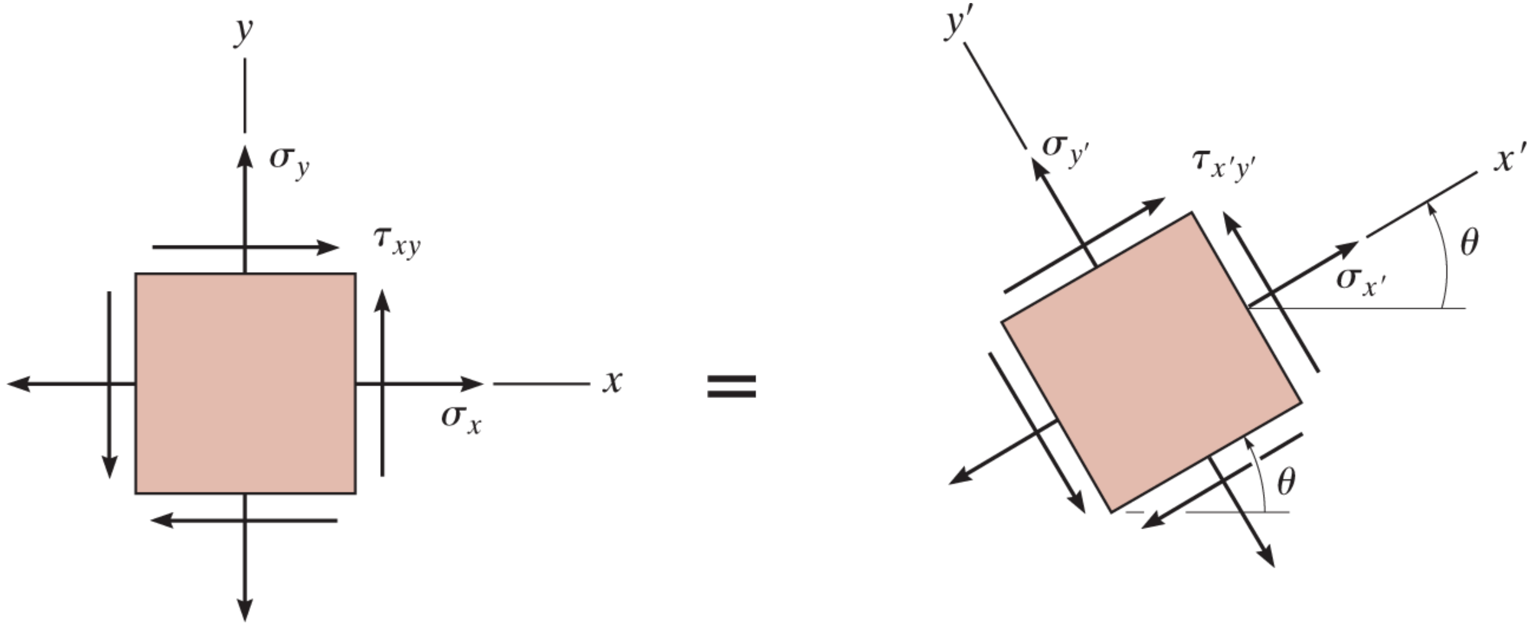
General state of stress



Plane stress

Plane-Stress Transformation

The state of plane stress at the point is uniquely represented by two normal stress components σ_x, σ_y and one shear stress component τ_{xy} acting on an element that has a specific orientation at the point.



Plane-Stress Transformation

Transformation		
	Force	Force component's magnitude and direction
	Stress	Stress component's magnitude and direction and the orientation of the area upon which each component acts

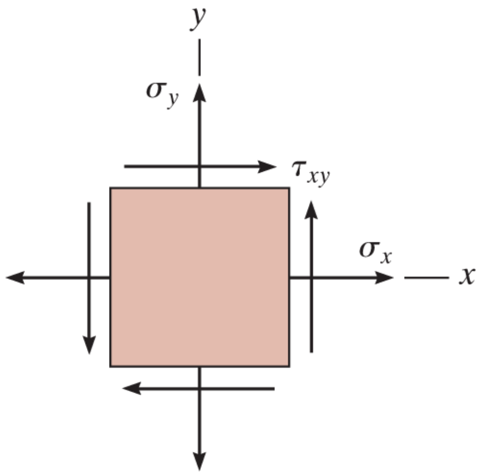
Plane-Stress Transformation

Sign Convention:

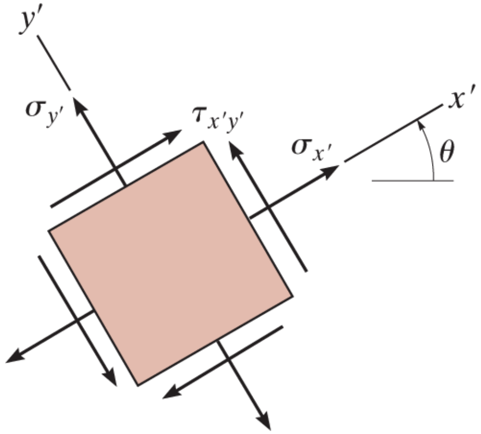
First we must establish a sign convention for the stress components.
To do this the $+x$ axes are used to define the outward normal from a side of the element. Then and σ_x are positive when they act in the positive x directions, and τ_{xy} are positive when they act in the positive y and directions.
The orientation of the plane on which the normal and shear stress components are to be determined will be defined by the angle θ, which is measured from the $+x$ axis to the $+x'$ axis,

Plane-Stress Transformation

If the state of stress at a point is known for a given orientation of an element of material.



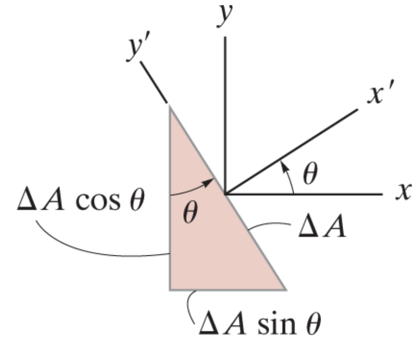
The state of stress in an element having some other orientation, can be determined using the following procedure:



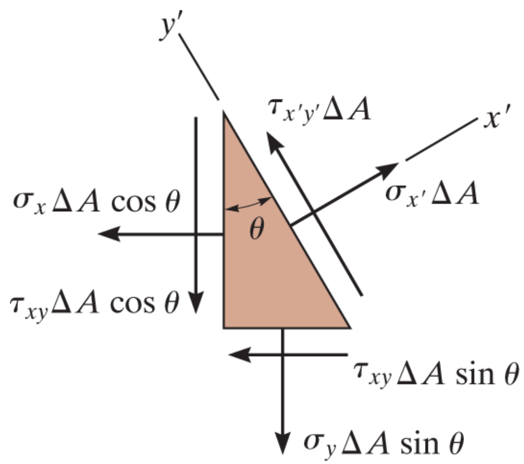
Plane-Stress Transformation

To determine the normal and shear stress components $\sigma_{x'}$ and $\tau_{x'y'}$ acting on x' the face of the element, section the element.

If the sectioned area is ΔA then the adjacent areas of the segment will be: $\Delta A.\cos\theta$, and $\Delta A.\sin\theta$.

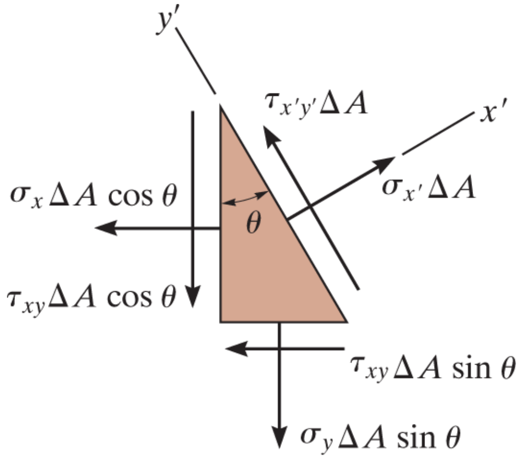


Draw the free-body diagram of the segment, which requires showing the forces that act on the segment. This is done by multiplying the stress components on each face by the area upon which they act.



Plane-Stress Transformation

Apply the force equations of equilibrium in the and directions. The area will cancel from the equations and so the two unknown stress components and can be determined.



$$\sum F_{x'} = 0 \Rightarrow$$

$$\sigma_{x'}.\Delta A - \sigma_x.\Delta A.\cos\theta.\cos\theta - \tau_{xy}.\Delta A.\cos\theta.\sin\theta - \sigma_y.\Delta A.\sin\theta.\sin\theta - \tau_{xy}.\Delta A.\sin\theta.\cos\theta = 0$$

$$\sigma_{x'} = \sigma_x.\cos^2\theta + \sigma_y.\sin^2\theta + 2\tau_{xy}.\cos\theta.\sin\theta$$

$$\sum F_{y'} = 0 \Rightarrow$$

$$\tau_{x'y'}.\Delta A + \sigma_x.\Delta A.\cos\theta.\sin\theta - \tau_{xy}.\Delta A.\cos\theta.\cos\theta - \sigma_y.\Delta A.\sin\theta.\cos\theta + \tau_{xy}.\Delta A.\sin\theta.\sin\theta = 0$$

$$\tau_{x'y'} = -\left(\sigma_x - \sigma_y\right)\cos\theta.\sin\theta + \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right)$$

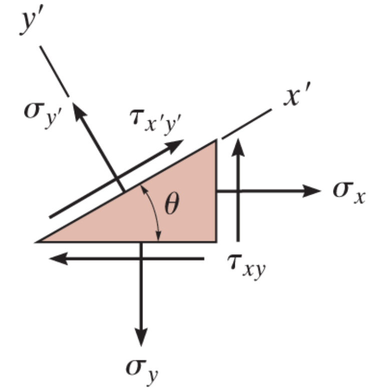
Plane-Stress Transformation

$$\left.\begin{aligned} \sin 2\theta &= 2\sin\theta\cos\theta \\ \sin^2\theta &= (1-\cos 2\theta)/2 \\ \cos^2\theta &= (1+\cos 2\theta)/2 \\ \cos 2\theta &= \cos^2\theta - \sin^2\theta \end{aligned}\right\} in \begin{cases} \sigma_{x'} = \sigma_x.\cos^2\theta + \sigma_y.\sin^2\theta + 2\tau_{xy}.\cos\theta.\sin\theta \\ \tau_{x'y'} = -(\sigma_x - \sigma_y)\cos\theta.\sin\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) \end{cases}$$

$$\Rightarrow \begin{cases} \sigma_{x'} = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\ \tau_{x'y'} = -\dfrac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \end{cases}$$

Plane-Stress Transformation

Consider a segment of the element and follow the same procedure. Here, however, the shear stress does not have to be determined if it was previously calculated, since it is complementary, that is, it must have the same magnitude on each of the four faces of the element.



$$\sum F_{y'} = 0 \Rightarrow$$

$$\sigma_{y'}.\Delta A - \sigma_x.\Delta A.\sin\theta.\sin\theta + \tau_{xy}.\Delta A.\sin\theta.\cos\theta - \sigma_y.\Delta A.\cos\theta.\cos\theta + \tau_{xy}.\Delta A.\cos\theta.\sin\theta = 0$$

$$\sigma_{y'} = \sigma_x.\sin^2\theta + \sigma_y.\cos^2\theta + 2\tau_{xy}.\cos\theta.\sin\theta$$

$$\sigma_{y'} = \sigma_x.(1-\cos 2\theta)/2 + \sigma_y.(1+\cos 2\theta)/2 - \tau_{xy}\sin 2\theta$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

Plane-Stress Transformation

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Principal Stresses and Maximum In-Plane Shear Stress

It can be seen that the magnitudes of $\sigma_{x'}$ and $\tau_{x'y'}$ depend on the angle of inclination θ of the planes on which these stresses act. In engineering practice it is often important to determine the orientation of the element that causes the normal stress to be a maximum and a minimum and the orientation that causes the shear stress to be a maximum.

Principal Stresses and Maximum In-Plane Shear Stress

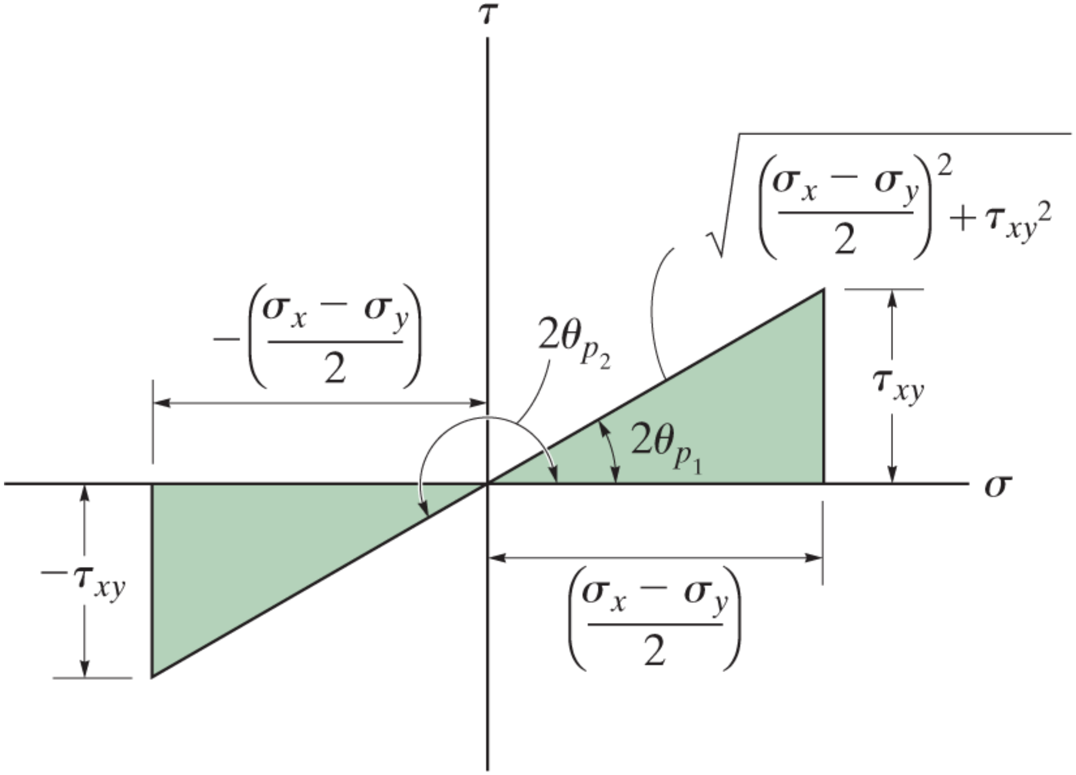
In-Plane Principal Stresses

$$\left.\begin{aligned} \frac{d\sigma_{x'}}{d\theta} &= -2\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + 2\tau_{xy}\cos 2\theta \\ \frac{d\sigma_{x'}}{d\theta} &= 0 \Rightarrow \theta = \theta_P \end{aligned}\right\} \Rightarrow \tan 2\theta_P = \frac{\tau_{xy}}{\left(\sigma_x - \sigma_y\right)/2}$$

Principal Stresses and Maximum In-Plane Shear Stress

In-Plane Principal Stresses

The solution has two roots $2\theta_{P1}$, $2\theta_{P2}$ where $2\theta_{P2}=2\theta_{P1}+\pi$ and then $\theta_{P2}=\theta_{P1}+\pi/2$



Principal Stresses and Maximum In-Plane Shear Stress

In-Plane Principal Stresses

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \left(\frac{\sigma_x - \sigma_y}{2} \frac{\frac{\sigma_x - \sigma_y}{2}}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}} + \tau_{xy} \frac{\tau_{xy}}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}} \right)$$

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Principal Stresses and Maximum In-Plane Shear Stress

In-Plane Principal Stresses

$$\tau_P = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\tau_P = -\frac{\sigma_x - \sigma_y}{2}\frac{\tau_{xy}}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}} + \tau_{xy}\frac{\frac{\sigma_x - \sigma_y}{2}}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}} = 0$$

Principal Stresses and Maximum In-Plane Shear Stress

Maximum In-Plane Shear Stress.

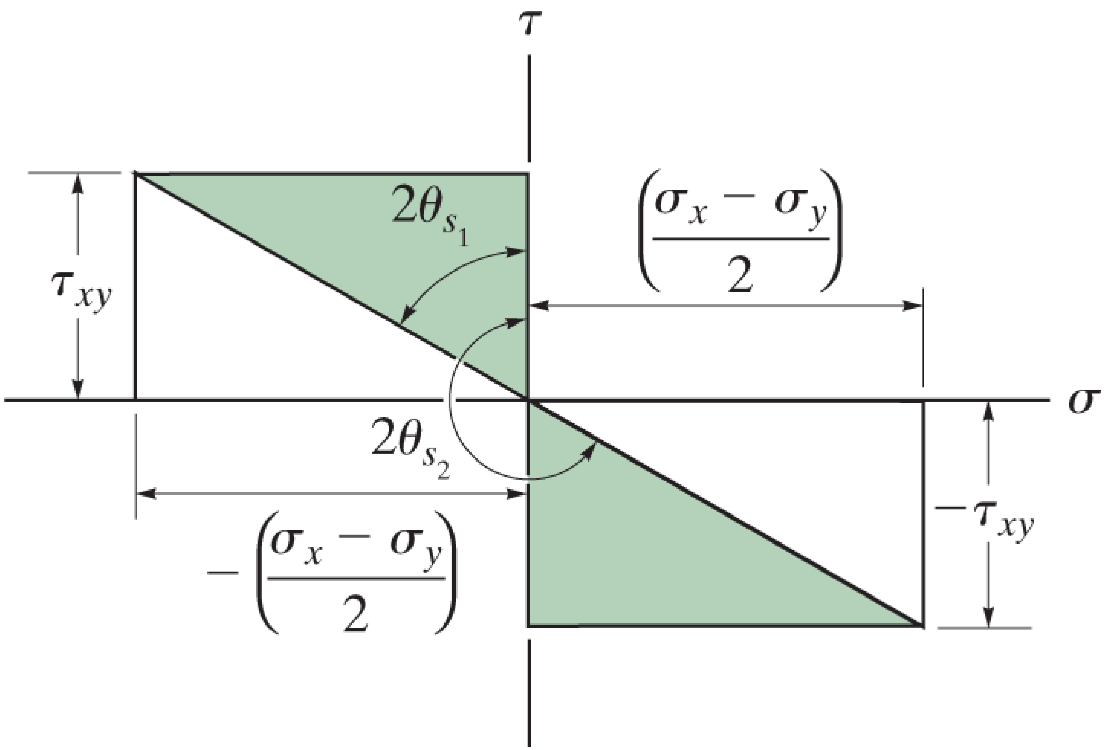
The orientation of an element that is subjected to maximum shear stress on its sides can be determined by taking the derivative of $\tau_{x'y'}$ Eq. with respect to θ and setting the result equal to zero. This gives

$$\left.\begin{aligned} \tau_{x'y'} &= \frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \\ \frac{d\tau_{x'y'}}{d\theta} &= 0 \Rightarrow \theta = \theta_S \end{aligned}\right\} \Rightarrow \tan 2\theta_S = \frac{-\left(\sigma_x - \sigma_y\right)/2}{\tau_{xy}}$$

Principal Stresses and Maximum In-Plane Shear Stress

Maximum In-Plane Shear Stress.

The solution has two roots $2\theta_{S1}$, $2\theta_{S2}$ where $2\theta_{S2}=2\theta_{S1}+\pi$ and then $\theta_{S2}=\theta_{S1}+\pi/2$



Principal Stresses and Maximum In-Plane Shear Stress

Maximum In-Plane Shear Stress.

$$\left.\begin{aligned} \tan 2\theta_P &= \frac{\tau_{xy}}{\left(\sigma_x - \sigma_y\right)/2} \\ \tan 2\theta_S &= \frac{-\left(\sigma_x - \sigma_y\right)/2}{\tau_{xy}} \end{aligned}\right\} \Rightarrow$$

$\tan 2\theta_S$ is the negative reciprocal of $\tan 2\theta_P$

$$\Downarrow$$

an element subjected to maximum shear stress will be 45° from the position of an element that is subjected to the principal stress.

Principal Stresses and Maximum In-Plane Shear Stress

Maximum In-Plane Shear Stress.

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\tau_{\substack{\max \\ \text{in-plane}}} = -\frac{\sigma_x - \sigma_y}{2}\,\frac{-\dfrac{\sigma_x - \sigma_y}{2}}{\sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}} + \tau_{xy}\,\frac{\tau_{xy}}{\sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

$$\tau_{\substack{\max \\ \text{in-plane}}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Principal Stresses and Maximum In-Plane Shear Stress

Maximum In-Plane Shear Stress.

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\frac{\tau_{xy}}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}} - \tau_{xy}\frac{\frac{\sigma_x - \sigma_y}{2}}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

$$\sigma_{x'} = \sigma_{y'} = \sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}$$

Mohr's Circle Plane Stress

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x'y'}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

Mohr's Circle Plane Stress

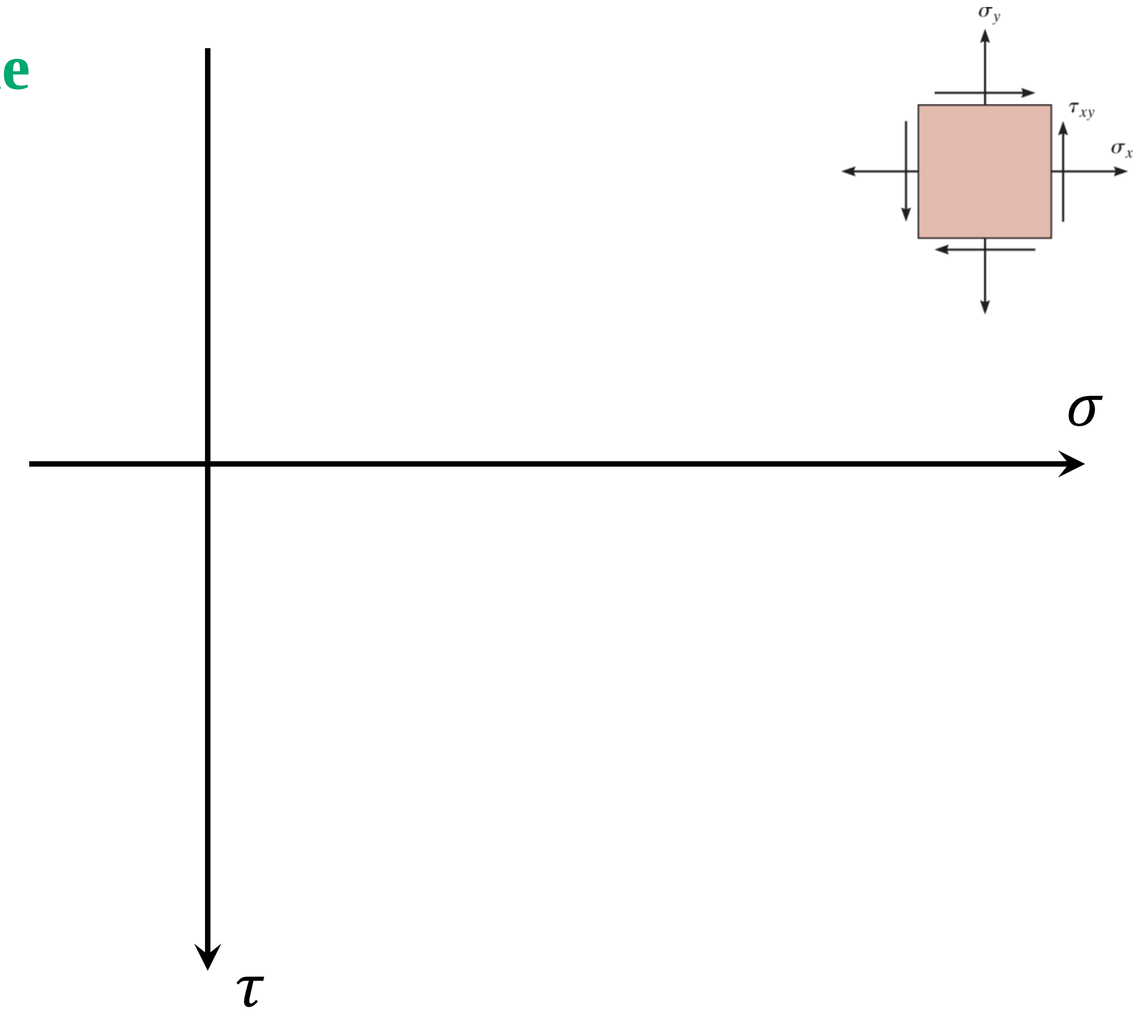
$$\left(\sigma_{x'} - \sigma_{avg}\right)^2 + \tau_{x'y'}^2 = R^2 \begin{cases} \sigma_{avg} = \left(\sigma_x + \sigma_y\right)/2 \\ R = \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \end{cases}$$

Mohr’s Circle Plane Stress

Construction of the Circle

If we establish coordinate axes, $\sigma_{x'}$ positive to the right and $\tau_{x'y'}$ positive downward, and then plot the equation

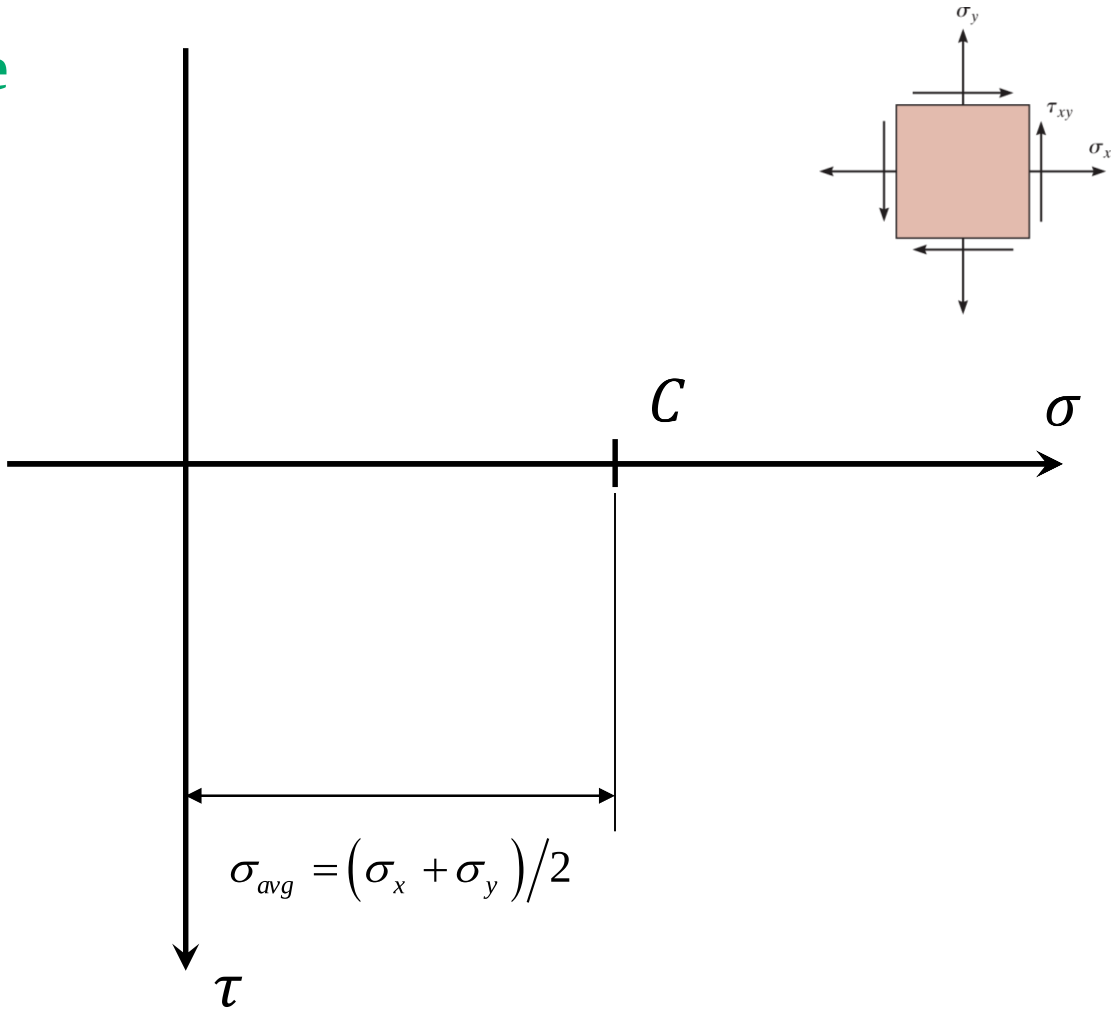
$$\left(\sigma_{x'} - \sigma_{avg}\right)^2 + \tau_{x'y'}^2 = R^2$$



Mohr's Circle Plane Stress

Construction of the Circle

Using the positive sign convention for σ_x , σ_y and τ_{xy}, plot the center of the circle C, which is located on the axis at a distance C:$\left(\sigma_{avg}, 0\right)$ from the origin.

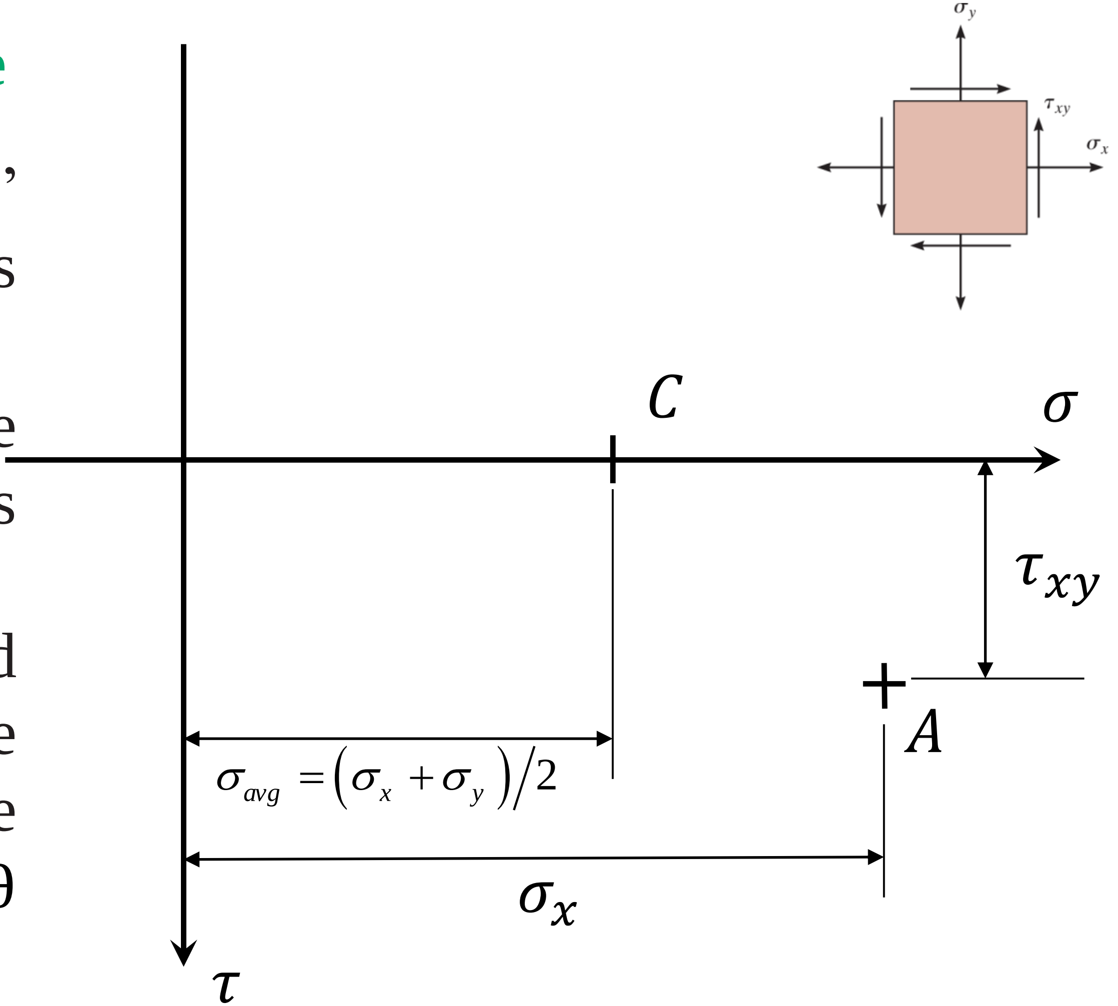


Mohr's Circle Plane Stress

Construction of the Circle

Plot the "reference point" A having coordinates $A(\sigma_x, \tau_{xy})$.

This point represents the normal and shear stress components on the element's right-hand vertical face, and since the x axis coincides with the x' axis, this represents θ = 0

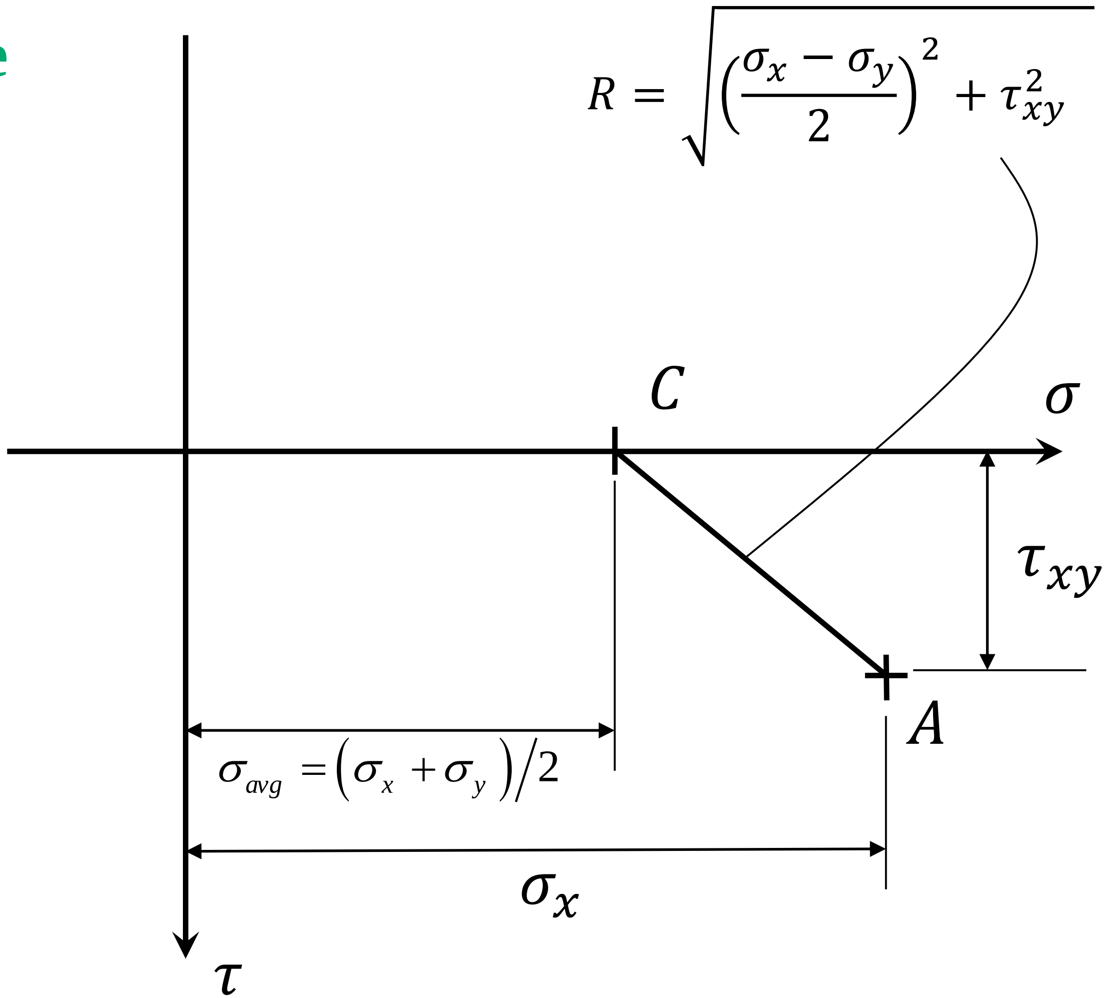


Mohr's Circle Plane Stress

Construction of the Circle

Connect point A with the center C of the circle and determine CA by trigonometry. This distance represents the radius R of the circle,

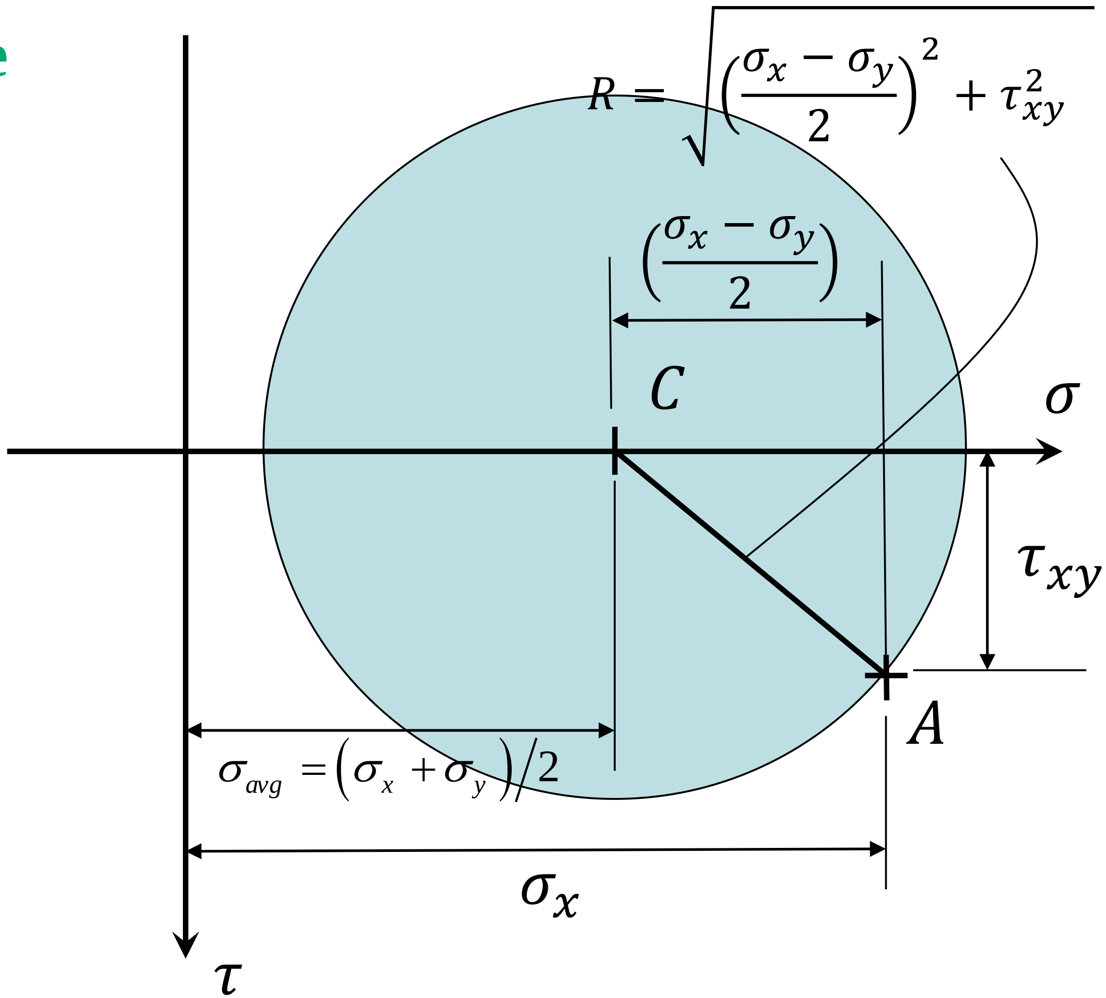
$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$



Mohr’s Circle Plane Stress

Construction of the Circle

Once R has been determined, sketch the circle

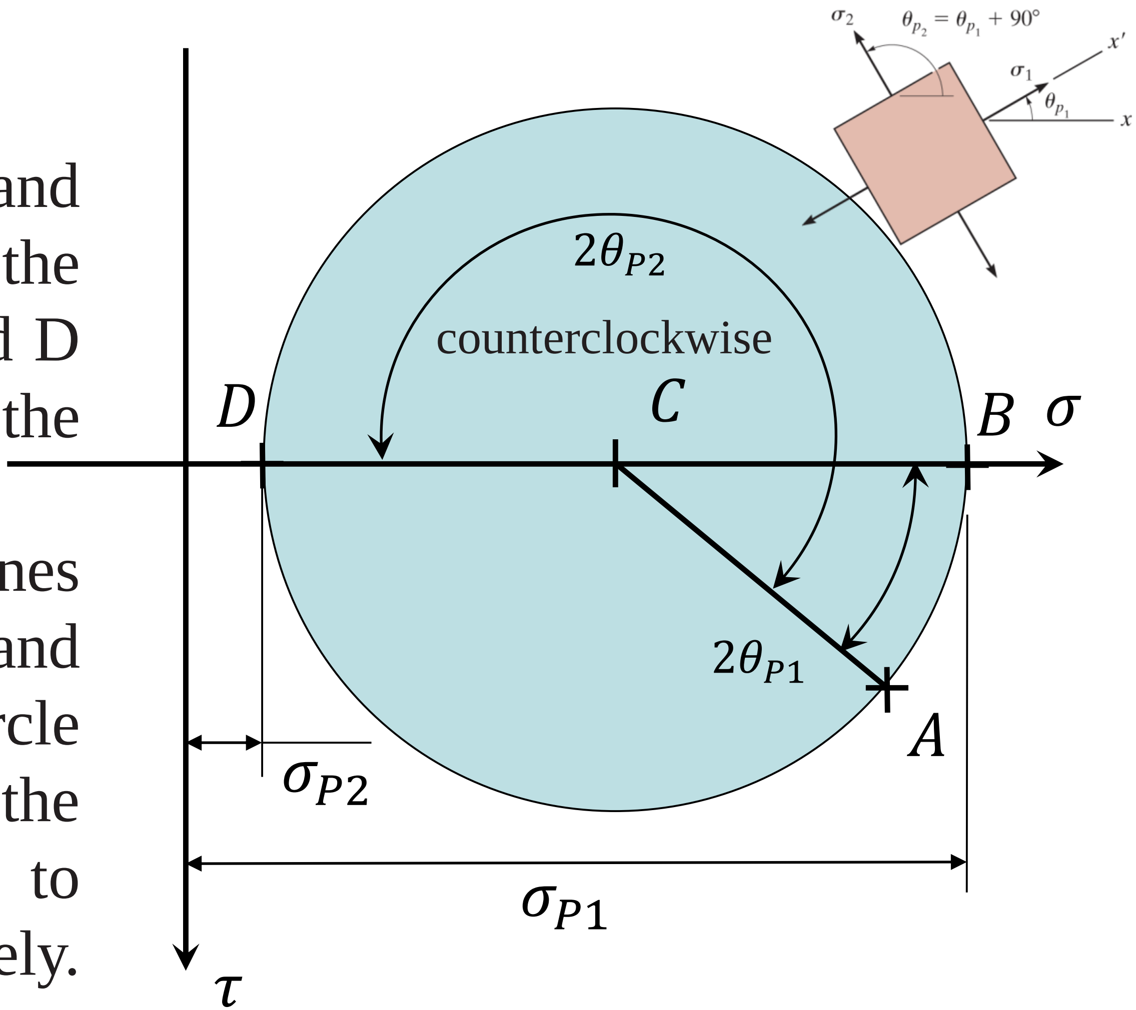


Mohr’s Circle Plane Stress

Principal Stress

The principal stresses σ_{P1} and $\sigma_{P2} (\sigma_{P1} \geq \sigma_{P2})$ are the coordinates of points B and D where the circle intersects the axis σ, i.e., where $\tau = 0$

These stresses act on planes defined by angles θ_{P1} and θ_{P2}, represented on the circle by angles measured from the radial reference line CA to lines CB and CD, respectively.

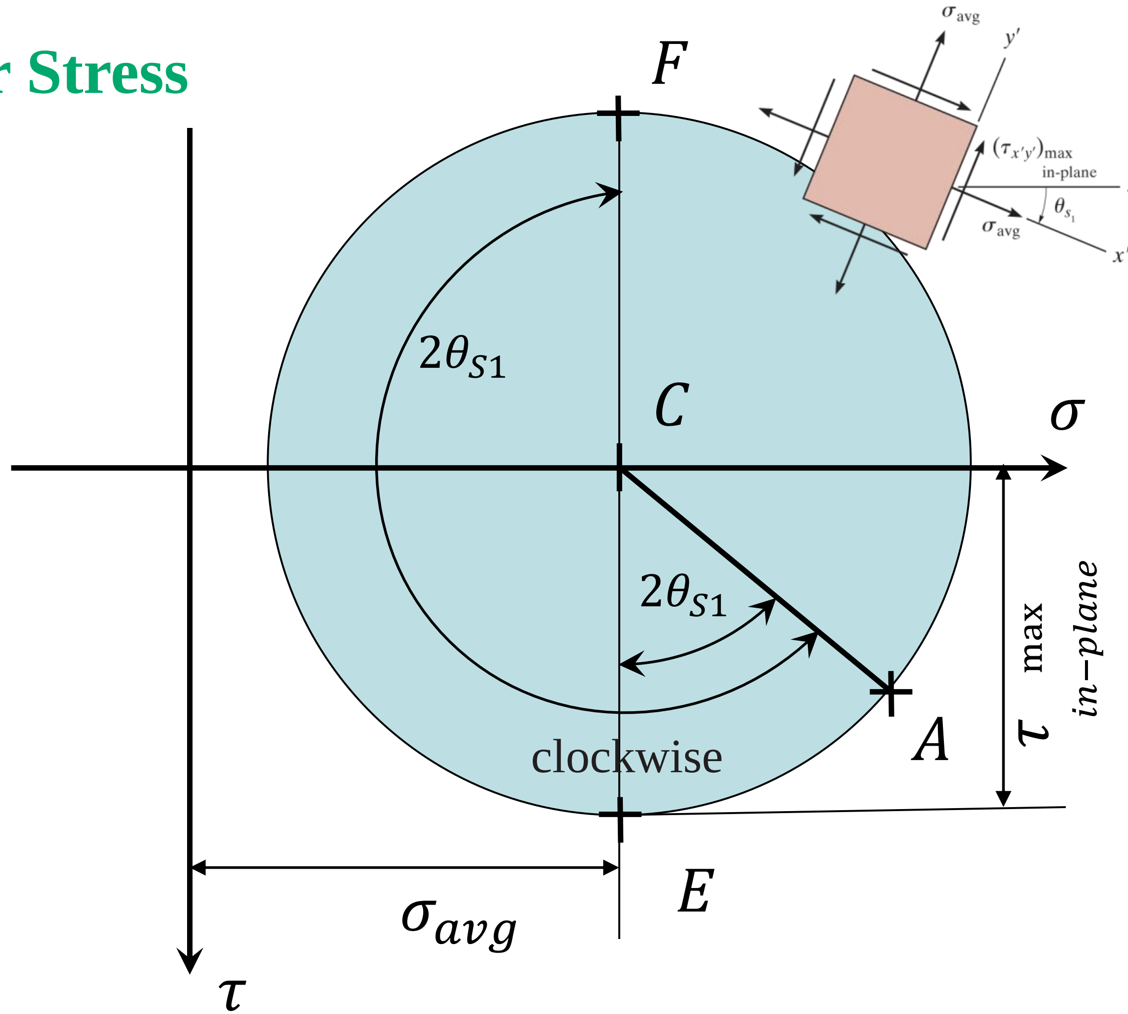


Mohr’s Circle Plane Stress

Maximum In-Plane Shear Stress

The average normal stress and maximum in-plane shear stress components are determined from the circle as the coordinates of either point E or F.

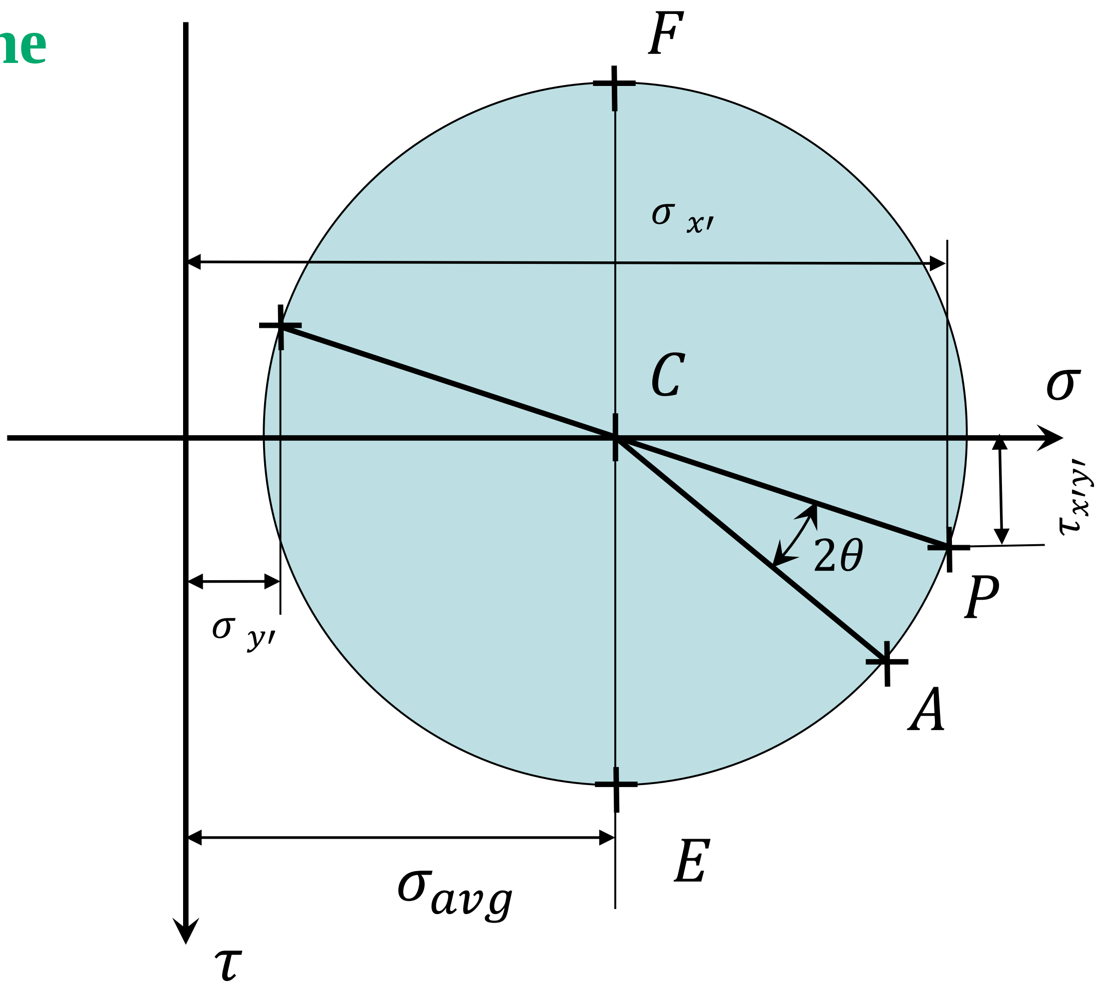
the angles and give the orientation of the planes that contain these components.



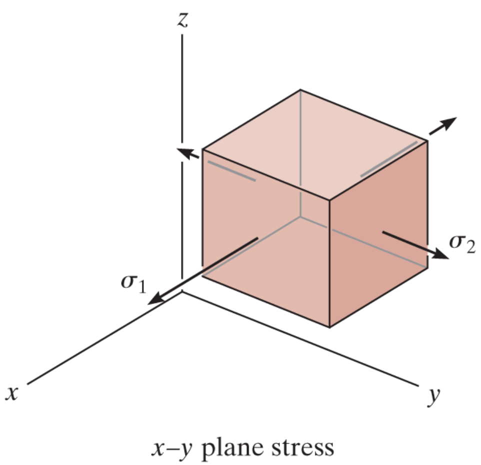
Mohr's Circle Plane Stress

Stresses on Arbitrary Plane

To locate P, the known angle θ (in this case counterclockwise), must be measured on the circle in the same direction 2θ, from the radial reference line CA to the radial line CP

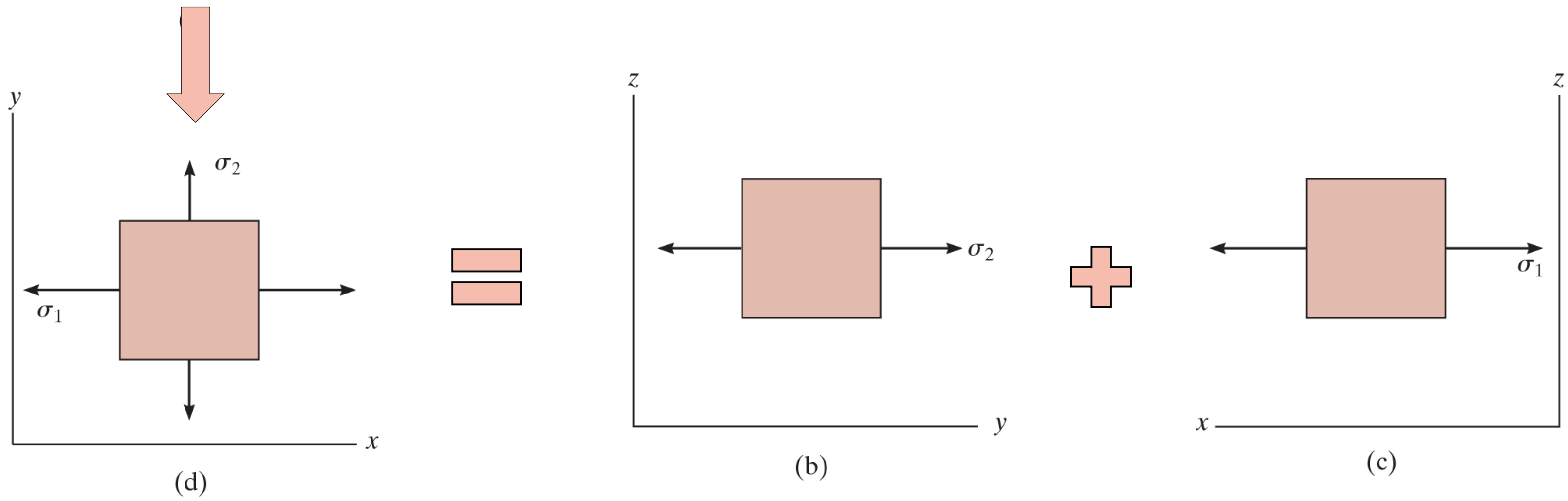


Absolute maximum shear stress



σ_1 and σ_2 have the same sign

material to be subjected to the in-plane principal stresses σ_1 and σ_2 where both of these stresses are tensile



Absolute maximum shear stress

For all three circles, it is seen that although the maximum in-plane shear stress is

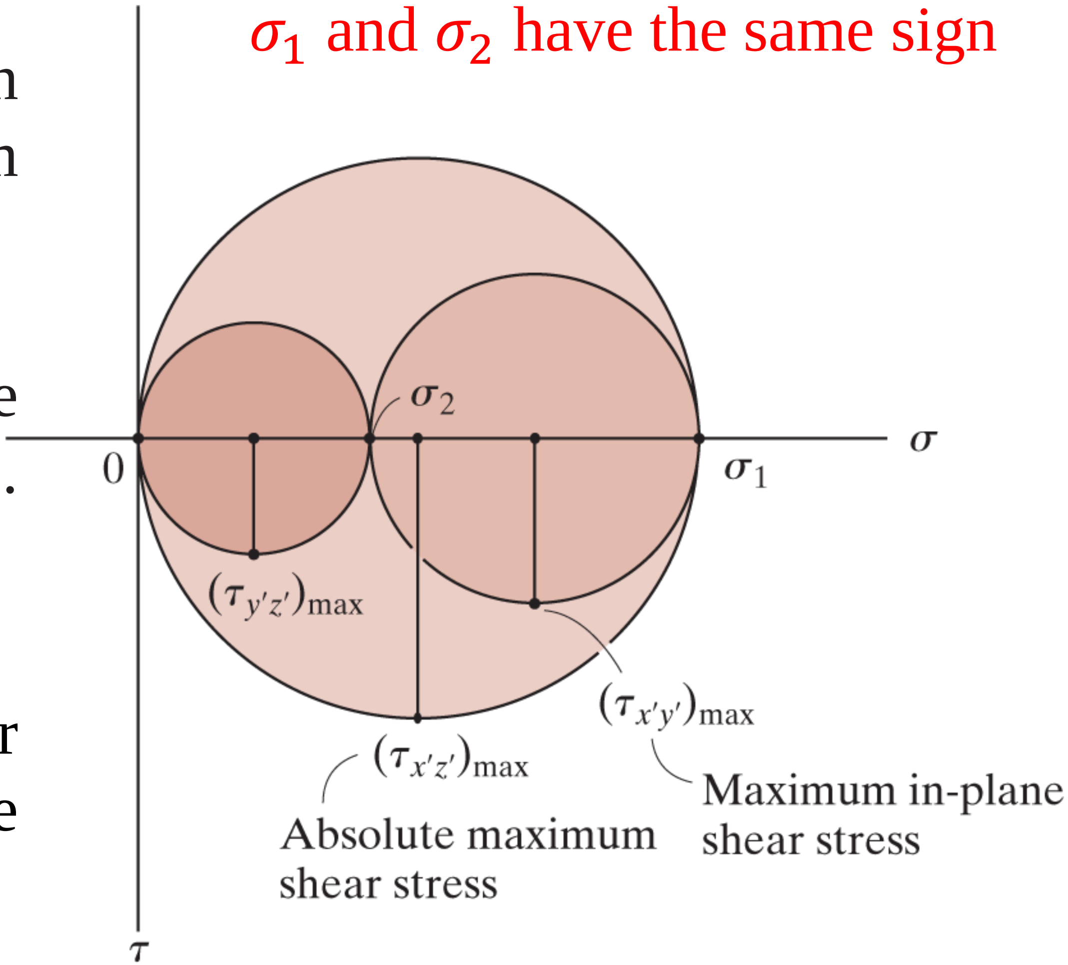
$$\tau_{x'y'} = (\sigma_1 - \sigma_2)/2$$

This value is *not* the absolute maximum shear stress. Instead, from the figure

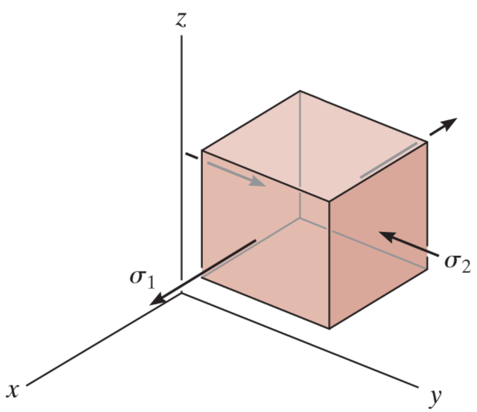
$$\tau_{\substack{abs\\max}} = \sigma_1/2$$

The absolute maximum shear stress will occur out of the plane

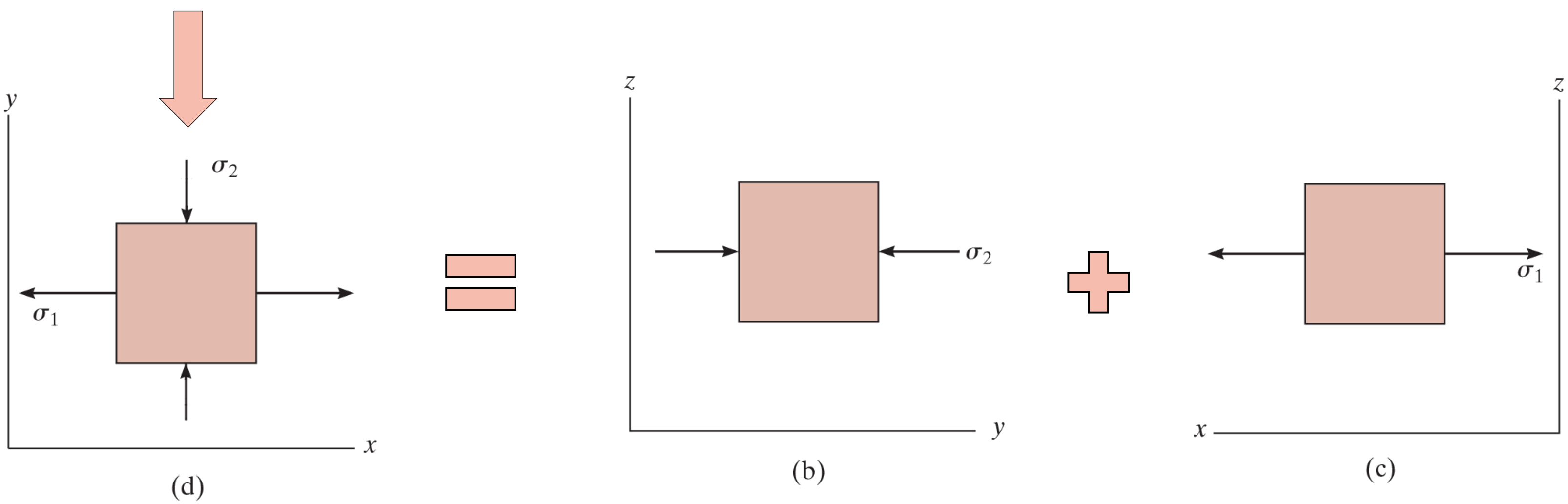
σ_1 and σ_2 have the same sign



Absolute maximum shear stress



x–y plane stress



σ_1 and σ_2 have the opposite signs

material to be subjected to the in-plane principal stresses σ_1 is tensile and σ_2 is compression.

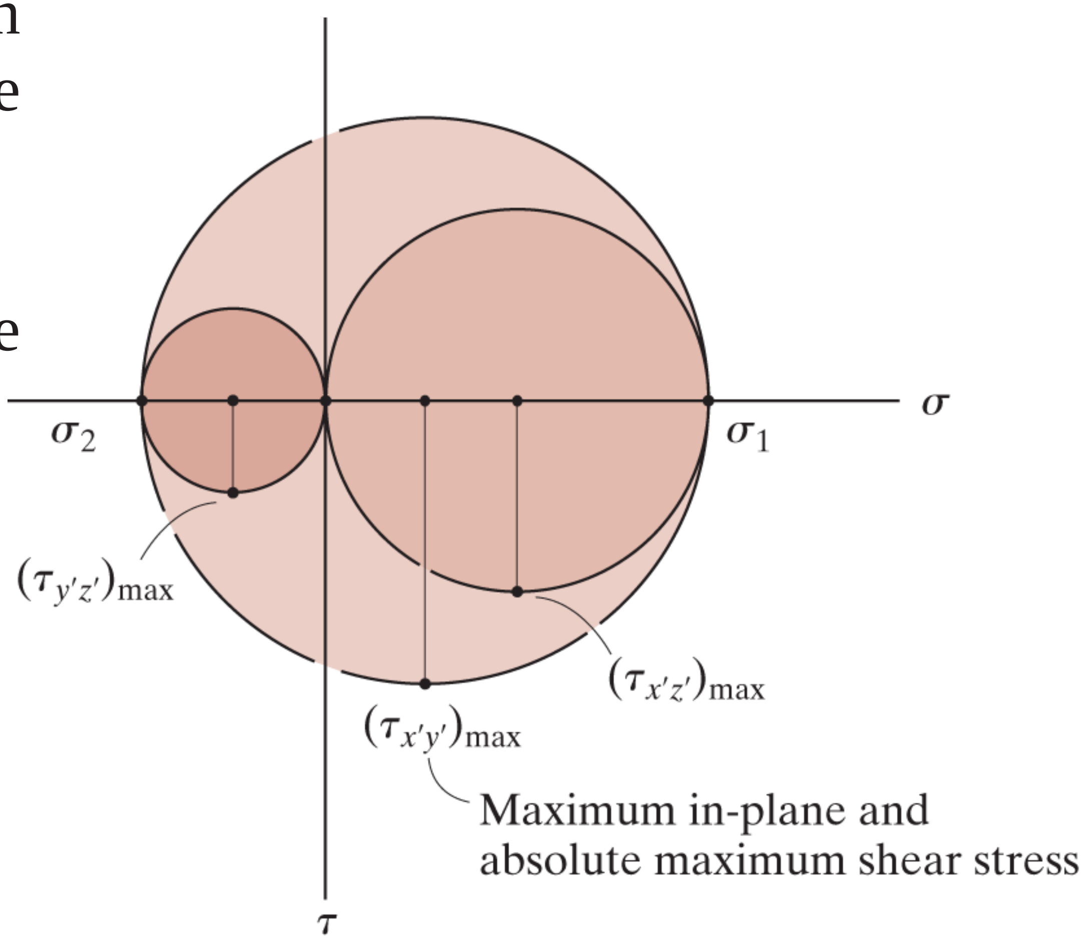
Absolute maximum shear stress

σ_1 and σ_2 have the opposite signs

For all three circles, it is seen that the maximum in-plane shear stress is

$$\tau_{x'y'} = (\sigma_1 - \sigma_2)/2$$

Equal to the absolute maximum shear stress.



Absolute maximum shear stress